

# Triangular spectrum of some triangulated categories

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## 1 Introduction

This paper studies the prime spectrum of two tensor triangulated categories. Triangulated categories have been one of the most influential objects in mathematics. Introduced by Grothendieck and Verdier to study Serre duality in a relative setting, this idea was soon developed by Verdier and Illusie who studied the derived category of the abelian category of coherent sheaves, and the triangulated category of perfect complexes respectively. Slowly the abstract homological construction of triangulated categories permeated into other subjects like topology, modular representation theory and even Kasparov's KK theory. Balmer's paper [2] gives a nice summary of the elegant history.

In algebraic geometry, triangulated categories mostly appear as the derived category of the abelian category of coherent sheaves on a variety and as the category of perfect complexes on a variety. The later category, as was observed by Neeman [14], are just the compact objects of the derived category of the abelian category of quasi-coherent sheaves (in case the scheme is quasi compact and separated). From now on we shall call the derived category of the category of coherent sheaves to be the derived category of the variety. Gabriel [5] and Rosenberg [16] proved that the category of quasi coherent sheaves completely determine the underlying variety. Bondal and Orlov [4] proved that a smooth variety can be reconstructed from the derived category of coherent sheaves provided that either the canonical bundle or the anti-canonical bundle is ample. But the ampleness condition here is crucial, as Mukai [11] gave an example of two nonisomorphic varieties whose derived categories are equivalent.

Balmer [2] proved that in addition to the triangulated structure on a derived category, if we also consider the tensor structure induced by the tensor structure in the category of coherent sheaves, we have enough information to reconstruct the variety. He gave a method to reconstruct, by constructing “the Spec” of the tensor triangulated category. The definition of Spec is quite general and applies to any tensor triangulated category. One question that naturally arises is how good is Spec as an invariant of the tensor triangulated category? It turns out that there do exist pairs of tensor triangulated categories which have isomorphic Specs (isomorphic as ringed spaces). We give two such examples. This raises the

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question of whether one can define a finer geometric invariant. Some possible answers are discussed in the first author's thesis with HBNI.

In section 2, we recall the definition of Spec. We also recall some facts about  $G$  sheaves and prove some lemmas which shall be useful in the next section.

In section 3 we compute the Spec of the derived category of the abelian category of coherent  $G$ -equivariant sheaves on some smooth quasi-projective scheme  $X$ . Since the scheme is quasi projective there exists an orbit space, see [12], which we denote as  $X/G$ . As  $G$  is a finite group and hence we get a finite map  $\pi : X \rightarrow X/G$  which is also a perfect morphism. Recall that a  $G$  equivariant or  $G$  linearized sheaf is defined as follows

**Definition 1.1.** A  $G$ -sheaf (or  $G$ -equivariant sheaf or an equivariant sheaf with respect to the group  $G$ ) on  $X$  is a sheaf  $\mathcal{F}$  together with isomorphisms  $\rho_g : \mathcal{F} \rightarrow g^* \mathcal{F}$  for all  $g \in G$  such that following diagram

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\rho_h} & h^* \mathcal{F} & \xrightarrow{h^* \rho_g} & h^* g^* \mathcal{F} \\ & \searrow \rho_{gh} & & & \parallel \\ & & & & (gh)^* \mathcal{F} \end{array}$$

is commutative for any pair  $g, h \in G$ . A  $G$ -sheaf is a pair  $(\mathcal{F}, \rho)$ .

The category of coherent  $G$ -sheaves is denoted as  $\mathcal{Coh}^G(X)$  and for simplicity we denote by  $\mathcal{D}^G(X)$ , the bounded derived category of coherent  $G$ -sheaves. Consider the affine map  $\pi : X \rightarrow X/G$ . Then  $\mathcal{D}^G(X)$  admits a functor from  $\mathcal{D}^{per}(X/G)$ ,

$$\pi^* : \mathcal{D}^{per}(X/G) \rightarrow \mathcal{D}^G(X).$$

Since we consider only quasi projective varieties therefore the perfect complexes are nothing but bounded complexes of vector bundles.

We prove the following theorem.

**Theorem 1.2.** *Assume that the scheme  $X$  is a smooth quasi projective variety over a field  $k$  of characteristic  $p$  with an action of a finite group  $G$ . Assume that the order of  $G$  is coprime to  $p$ . The induced map*

$$\mathrm{Spec}(\pi^*) : \mathrm{Spec}(\mathcal{D}^G(X)) \rightarrow \mathrm{Spec}(\mathcal{D}^{per}(X/G))$$

*is an isomorphism of locally ringed spaces.*

The proof involves some computation using results from representation theory.

Superschemes, studied by Manin and Deligne (see for example [10]), are also an important object of study in modern algebraic geometry, specially due to applications in physics. The following definition of split superscheme is given in [Pg. 84-85, Manin[9]].

**Definition 1.3.** 1. A ringed space  $(X, \mathcal{O}_X)$  is called *superspace* if the ring  $\mathcal{O}_X(U)$  associated to any open subset  $U$  is supercommutative and each stalk is local ring. A *superspace* is called *superscheme* if in addition the ringed space  $(X, \mathcal{O}_{X,0})$  is a scheme and  $\mathcal{O}_{X,1}$  is a coherent sheaf over  $\mathcal{O}_{X,0}$  (where the subscript 0 denotes the even part and the subscript 1 denotes the odd part). We shall denote by  $J_X$  the ideal sheaf generated by  $\mathcal{O}_{X,1}$  inside  $\mathcal{O}_X$ .

2. A superscheme  $(X, \mathcal{O}_X)$  is called *split* if the graded sheaf  $Gr\mathcal{O}_X$  with mod 2 grading is isomorphic as a locally superringed sheaf to  $\mathcal{O}_X$ . Here the graded sheaf

$$Gr\mathcal{O}_X := \oplus_{i \geq 0} J_X^i / J_X^{i+1} \text{ where } J_X^0 := \mathcal{O}_X.$$

Manin has also given example of superschemes which are not split superschemes. An important example of a split superscheme is super projective space  $\mathbb{P}^{n|m}$ . We consider the triangulated category  $\mathcal{D}^{per}(X)$  of “ perfect complexes ” (the definition being modified appropriately in the super setting) on this superscheme.

**Theorem 1.4.** *Let  $X$  be a split superscheme. Let  $X_0 = (X, \mathcal{O}_{X,0})$  be the 0-th part of this superscheme. Here  $X_0$  is by definition a scheme. Then we have an isomorphism of locally ringed spaces*

$$f : X_0 \rightarrow \text{Spec}(\mathcal{D}^{per}(X)).$$

The proof of homeomorphism adapts the classification of thick tensor ideals due to Thomason[17] as demonstrated by Balmer[2]. Again, following Balmer[2] we use the generalized localization theorem of [Theorem 2.1, Neeman[14]] to finish the proof.

**Acknowledgement :** We would like to take this opportunity to thank Prof. Kapil Paranjape and Prof. V. Srinivas for their encouragement and insightful comments.

## 2 Preliminaries

In this section we shall recall various basic definitions and facts which are used explicitly or implicitly later.

### 2.1 Categorical definitions

As we are borrowing many definitions and results from Balmer’s papers[1][2] so we shall work only with an essentially small categories i.e. categories equivalent to a small category. We recall first some basic definitions,

**Definition 2.1** (Semi simple abelian category). An abelian category is called *semisimple* if every short exact sequence splits.

**Definition 2.2** (Triangulated category). An additive category  $\mathcal{D}$  with a functorial isomorphism  $T$ , (called *translation* or *shift*,) and a collection of sextuple  $(a, b, c, f, g, h)$  with objects  $a, b, c$  and morphisms  $f, g, h$ , called *distinguished triangles*, satisfying certain axioms, (cf. [19][8],) is called *triangulated category*. Traditionally the image of any object, say  $a$ , via functor  $T^i$  is denoted as  $a[i]$  and a distinguished triangle is denoted in a similar way as short exact sequences:  $a \rightarrow b \rightarrow c \rightarrow a[1]$ .

An additive functor  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  between two triangulated categories  $\mathcal{D}_1$  and  $\mathcal{D}_2$  is called a *triangulated functor* if it commutes with the translation functor i.e.  $F \circ T = T \circ F$  and takes distinguished triangles to distinguished triangles, i.e.  $F(a) \rightarrow F(b) \rightarrow F(c) \rightarrow F(a)[1]$  is distinguished for every distinguished triangles  $a \rightarrow b \rightarrow c \rightarrow a[1]$ .

*Example 2.3.* Let  $\mathcal{A}$  be an abelian category and  $K^*(\mathcal{A})$  (resp.  $\mathcal{D}^*(\mathcal{A})$ ), for  $(* = -, + \text{ or } b)$ , be the homotopy (resp. derived) category of an abelian category  $\mathcal{A}$ . Then both additive categories are triangulated categories, see [6] for proof. In particular we are interested in the cases when  $\mathcal{A} = \text{Coh}^G(X)$  for some variety  $X$  with an action of some finite group  $G$ ; see subsection 2.3 for more details. When group  $G$  is trivial then  $\mathcal{A}$  is an abelian category of coherent sheaves on variety  $X$ . Another class of examples which we shall consider later comes from an abelian categories  $\mathcal{A} = \text{Coh}(\mathcal{O}_X)$  for some split superscheme  $X$ .

*Example 2.4.* The category  $\mathcal{D}^{per}$  of perfect complexes on a scheme is a triangulated category. See [18] for definitions.

## 2.2 Triangular spectrum

In this section we shall recall some definitions and results from Balmer's papers [1] and [2]. Suppose  $\mathcal{D}$  is an essentially small triangulated category

**Definition 2.5.** A *tensor triangulated category* is a triple  $(\mathcal{D}, \otimes, 1)$  consisting of a triangulated category with symmetric monoidal bifunctor which is exact in each variable. The unit is denoted by 1 (or Id).

**Definition 2.6.** A *thick tensor ideal*  $\mathcal{A}$  of  $\mathcal{D}$  is a full sub category containing 0 and satisfying the following conditions:

- (a)  $\mathcal{A}$  is *triangulated*: if any two terms of a distinguished triangle are in  $\mathcal{A}$  then third term is also in  $\mathcal{A}$ . In particular direct sum of any two objects of  $\mathcal{A}$  is again in  $\mathcal{A}$  and this we refer as an **additivity**.
- (b)  $\mathcal{A}$  is *thick*: If  $a \oplus b \in \mathcal{A}$  then  $a \in \mathcal{A}$ .
- (c)  $\mathcal{A}$  is *tensor ideal*: if  $a$  or  $b \in \mathcal{A}$  then  $a \otimes b \in \mathcal{A}$ .

If  $\mathcal{E}$  is any collection of  $\mathcal{D}$  then we shall denote by  $\langle \mathcal{E} \rangle$  the smallest thick tensor ideal generated by this subset in  $\mathcal{D}$ .

Now we shall give an explicit description of a thick tensor ideal generated by some collection  $\mathcal{E}$  in a tensor triangulated category. We first use some definitions from Bondal[3] here. Recall  $add(\mathcal{E})$  was defined as an additive category generated by  $\mathcal{E}$  and closed under taking shifts inside  $\mathcal{D}$ . Similarly define  $ideal(\mathcal{E})$  as a full sub category generated by objects of the form  $a \otimes x$  for each  $a \in \mathcal{D}$  and  $x \in \mathcal{E}$ . Therefore  $ideal(\mathcal{E})$  is closed under taking direct sum, shifts and tensoring with any object of  $\mathcal{D}$ . Recall the operation defined on sub categories i.e.  $\mathcal{A} \star \mathcal{B}$  is the full sub category generated by objects  $x$  which fits in a distinguished triangle of the form

$$a \rightarrow x \rightarrow b \rightarrow a[1] \text{ with } a \in \mathcal{A} \text{ and } b \in \mathcal{B}.$$

This was observed by Bondal[3] et. al. that if  $\mathcal{A}$  and  $\mathcal{B}$  are closed under shifts and direct sums then  $\mathcal{A} \star \mathcal{B}$  is also closed under shifts and direct sums. Similarly we can see that if  $\mathcal{A}$  and  $\mathcal{B}$  are tensor ideal then  $\mathcal{A} \star \mathcal{B}$  is also tensor ideal. Take  $smd(\mathcal{A})$  to be the full subcategory generated by all direct summands of objects of  $\mathcal{A}$ . Now combining these two operations we can define a new operation on collections of subcategories as follows,

$$\mathcal{A} \diamond \mathcal{B} := smd(\mathcal{A} \star \mathcal{B}).$$

Using this operation we can define the full subcategories  $\langle \mathcal{E} \rangle^n$  for each non-negative integer as

$$\langle \mathcal{E} \rangle^n := \langle \mathcal{E} \rangle^{n-1} \diamond \langle \mathcal{E} \rangle^0 \text{ where } \langle \mathcal{E} \rangle^0 := \text{smd}(\text{ideal}(\mathcal{E})).$$

Now we can see following description of ideal generated by a collection  $\mathcal{E}$ ,

**Lemma 2.7.**  $\langle \mathcal{E} \rangle = \bigcup_{n \geq 0} \langle \mathcal{E} \rangle^n$ .

Proof of the above lemma follows from the fact that right hand side subcategory is a thick tensor ideal and contains every thick tensor ideal containing the collection  $\mathcal{E}$ .

**Definition 2.8.** (a) An additive functor,  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ , is called an *exact (or triangulated)* if it commutes with translation functor and takes distinguished triangle to a distinguished triangle.

(b) An exact functor,  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ , is called a *tensor functor* if there exists a natural isomorphism  $\eta(a, b) : F(a) \otimes F(b) \rightarrow F(a \otimes b)$  for objects  $a$  and  $b$  of  $\mathcal{D}_1$ .

(c) A tensor functor,  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ , is called *dominant* if  $\langle F(\mathcal{D}_1) \rangle = \mathcal{D}_2$ .

Note that every unital tensor functors is a dominant tensor functor.

**Definition 2.9.** A *prime ideal* of  $\mathcal{D}$  is a proper thick tensor ideal  $\mathcal{P} \subsetneq \mathcal{D}$  such that  $a \otimes b \in \mathcal{P} \implies a \in \mathcal{P} \text{ or } b \in \mathcal{P}$ . And *triangular spectrum* of  $\mathcal{D}$  is defined as set of all prime ideals, i.e.

$$\text{Spc}(\mathcal{D}) = \{\mathcal{P} \mid \mathcal{P} \text{ is a prime ideal of } \mathcal{D}\}.$$

The Zariski topology on this set is defined as follows: closed sets are of the form

$$Z(\mathcal{S}) := \{\mathcal{P} \in \text{Spc}(\mathcal{D}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\},$$

where  $\mathcal{S}$  is a family of objects of  $\mathcal{D}$ ; or equivalently we can define the open subsets to be of the form

$$U(\mathcal{S}) := \text{Spc}(\mathcal{D}) \setminus Z(\mathcal{S}).$$

In particular, we shall denote by

$$\text{supp}(a) := Z(\{a\}) = \{\mathcal{P} \in \text{Spc}(\mathcal{D}) \mid a \notin \mathcal{P}\},$$

the basic closed sets and similarly  $U(\{a\})$  denotes the basic open sets.

A collection of objects  $\mathcal{S} \subset \mathcal{D}$  is called a *tensor multiplicative family* of objects if  $1 \in \mathcal{S}$  and if  $a, b \in \mathcal{S} \implies a \otimes b \in \mathcal{S}$ .

We shall recall here the following lemma (Lemma 2.2 in Balmer's paper[2]) which we shall need later,

**Lemma 2.10.** *Let  $\mathcal{D}$  be a nontrivial tensor triangulated category and  $\mathcal{I} \subset \mathcal{D}$  be a thick tensor ideal. Suppose  $\mathcal{S} \subset \mathcal{D}$  is a tensor multiplicative family of objects s.t.  $\mathcal{S} \cap \mathcal{I} = \emptyset$ . Then there exists a prime ideal  $\mathcal{P} \in \text{Spc}(\mathcal{D})$  such that  $\mathcal{I} \subset \mathcal{P}$  and  $\mathcal{P} \cap \mathcal{S} = \emptyset$ .*

Balmer [2] had also proved the functoriality of  $\mathrm{Spc}$  on all essentially small tensor triangulated category with a morphism given by an unital tensor functors but it is not difficult to see that it is also true for an essentially small tensor triangulated categories with morphism given by a dominant tensor functor i.e. we have following result,

**Proposition 2.11.** *Given  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  a dominant tensor functor, the map  $\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{D}_2) \rightarrow \mathrm{Spc}(\mathcal{D}_1)$  defined as  $\mathcal{P} \mapsto F^{-1}(\mathcal{P})$  is well defined, continuous and for all objects  $a \in \mathcal{D}_1$ , we have  $\mathrm{Spc}(F)^{-1}(\mathrm{supp}(a)) = \mathrm{supp}(F(a))$  in  $\mathrm{Spc}(\mathcal{D}_2)$ .*

*This defines a contravariant functor  $\mathrm{Spc}(-)$  from the category of essentially small tensor triangulated categories with dominant tensor functors as morphisms to the category of topological spaces. So if  $F, G$  are two dominant tensor functors then  $\mathrm{Spc}(G \circ F) = \mathrm{Spc}(F) \circ \mathrm{Spc}(G)$ .*

*Proof.* (Similar to Balmer[1]) □

**Corollary 2.12.** *If a tensor functor  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  is an equivalence then every quasi-inverse functor of  $F$  is a dominant tensor functor. And also  $\mathrm{Spc}(F)$  is a homeomorphism.*

*Proof.* First observe that the continuous map  $\mathrm{Spc}(F)$  given by a dominant tensor functor is independent of natural isomorphism defining the tensor functor (recall definition 2.8). Now using functoriality of above proposition we have an homeomorphism whenever a quasi-inverse of  $F$  is an dominant tensor functor. Suppose  $G$  is a quasi-inverse of  $F$ . Since  $G \circ F \simeq \mathrm{Id}$ , the exact functor  $G$  is dominant. Suppose  $\eta : F \circ G \rightarrow \mathrm{Id}$  and  $\mu : G \circ F \rightarrow \mathrm{Id}$  are natural isomorphisms. Now we get a required natural isomorphism by composing as follows,

$$G(a) \otimes G(b) \xrightarrow{\mu^{-1}} GF(G(a) \otimes G(b)) = G(FG(a) \otimes FG(b)) \xrightarrow{G(\eta_a \otimes \eta_b)} G(a \otimes b).$$

Here we used a fact that  $G(\eta_a \otimes \eta_b)$  gives a natural transformation. □

Now we shall recall the definition of a structure sheaf defined on  $\mathrm{Spc}(\mathcal{D})$  as in Balmer [2].

**Definition 2.13.** For any open set  $U \subset \mathrm{Spc}(\mathcal{D})$ , let  $Z := \mathrm{Spc}(\mathcal{D}) \setminus U$  be a closed complement and let  $\mathcal{D}_Z$  be the thick tensor ideal of  $\mathcal{D}$  supported on  $Z$ . We denote by  $\mathcal{O}_{\mathcal{D}}$  the sheafification of following presheaf of rings:  $U \mapsto \mathrm{End}(1_U)$  where  $1_U \in \frac{\mathcal{D}}{\mathcal{D}_Z}$  is the image of the unit 1 of  $\mathcal{D}$  via the localisation map. And the restriction maps are defined using localisation maps in the obvious way. The sheaf of commutative ring  $\mathcal{O}_{\mathcal{D}}$  makes the topological space  $\mathrm{Spc}(\mathcal{D})$  a ringed space, which we shall denote by  $\mathrm{Spec}(\mathcal{D}) := (\mathrm{Spc}(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ .

The following theorem was proved in Balmer[2] which computes the spectrum for certain tensor triangulated categories.

**Theorem 2.14** (Balmer). *For  $X$  a topologically noetherian scheme,*

$$\mathrm{Spec}(\mathcal{D}^{per}(X)) \simeq X.$$

### 2.3 $G$ -sheaves

Throughout this section,  $k$  is field and  $G$  be a finite group whose order is coprime to the characteristic of  $k$ . Let  $X$  be a smooth quasi-projective variety over  $k$ , with an action of a finite group  $G$  i.e. there is a group homomorphism from  $G$  to the automorphism group of algebraic variety  $X$ . We say  $G$  acts freely on  $X$  if  $gx \neq x$  for any  $x \in X$  and any  $g \in G$  with  $g \neq e$ . Recall following general result proved in Mumford's Book [12] for the existence of group quotient,

**Theorem 2.15.** *Let  $X$  be an algebraic variety and  $G$  a finite group of automorphisms of  $X$ . Suppose that for any  $x \in X$ , the orbit  $Gx$  of  $x$  is contained in an affine open subset of  $X$ . Then there is a pair  $(Y, \pi)$  where  $Y$  is a variety and  $\pi : X \rightarrow Y$  a morphism, satisfying:*

1. *as a topological space,  $(Y, \pi)$  is the quotient of  $X$  for the  $G$ -action; and*
2. *if  $\pi_*(\mathcal{O}_X)^G$  denotes the subsheaf of  $G$ -invariants of  $\pi_*(\mathcal{O}_X)$  for the action of  $G$  on  $\pi_*(\mathcal{O}_X)$  deduced from 1, the natural homomorphism  $\mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)^G$  is an isomorphism.*

*The pair  $(Y, \pi)$  is determined up to an isomorphism by these conditions. The morphism  $\pi$  is finite, surjective and separable.  $Y$  is affine if  $X$  is affine.*

*If further  $G$  acts freely on  $X$ ,  $\pi$  is an étale morphism.*

In the remark after the proof, Mumford further showed that quasi-projective varieties always satisfies the hypothesis of above theorem. We denote this quotient space (if it exists) as  $X/G$ . For a variety  $X$  with a  $G$  action, and  $H \subset G$  a subgroup, let  $X^H$  be the subvariety of fixed points of  $H$ .

**Proposition 2.16.** *With the notation in the above paragraph,*

1.  *$X^H$  is a closed subvariety.*
2. *If  $H_1 \subseteq H_2$  are subgroups then we have a reverse inclusion  $X^{H_2} \subseteq X^{H_1}$*
3. *If  $Y$  is any  $G$ -invariant component of  $X$  then there exists an open subset of  $Y$  with free action of  $G/H$  for unique subgroup  $H$ . A  $G$ -invariant component is defined to be a minimal  $G$ -invariant subset of  $X$  with dimension equal to  $\dim X$ . Here dimension of an algebraic set is the maximum of the dimensions of its irreducible subsets.*
4. *If  $Y$  is any  $G$ -invariant algebraic subset of  $X$  then there exists the set of subgroups  $H_i$  for  $i = 1, \dots, r$  and open subsets  $U_i, i = 1, \dots, r$  s.t.  $G/H_i$  acts freely on open subsets  $U_i$  for  $i = 1, \dots, r$  of  $Y$ . Here  $r$  is the number of  $G$ -invariant components of  $Y$ . Also note that each open subsets  $U_i$  for  $i = 1, \dots, r$  are pairwise disjoint.*

*Proof of 1.* Since  $X^H = \cap_{h \in H} X^h$  where  $X^h$  is a fixed points of automorphism corresponding to  $h$  under the action. It is enough to prove that the invariant of any automorphism of a variety is a closed subset. Since diagonal map gives a closed embedding we can take intersection with closed subset given by graph of automorphism. Hence it gives a closed subset of  $X$ .

*Proof of 2.* It clearly follows from the formulae  $X^{H_i} = \cap_{h \in H_i} X^h$ .

*Proof of 3.* Since for any algebraic subset there exists the subgroup  $H$  s.t.  $G/H$  acts faithfully(or effectively). Assume that  $G$  acts faithfully on  $Y$ . Since for a



faithful action  $Y^H$  is a proper subset of  $Y$  for any nontrivial subgroup of  $G$ . Define open subset of  $Y$  as

$$U = Y - (\cup_{H \trianglelefteq G} Y^H)$$

where union on right side is over all nontrivial subgroups and now it is easy to see that  $G$  acts freely on open set  $U$ .

*Proof of 4.* Using 3., it is enough to prove that any algebraic subset can be uniquely written as union of  $G$ -invariant components of  $X$ . Since  $X$  is noetherian it will be finite union of irreducible closed subsets. Take finite set  $S$  of generic points of irreducible subsets of  $X$  with same dimension as  $X$ . Now the action of  $G$  on  $X$  induces the action on finite set  $S$  as an automorphism of  $X$  will take any irreducible subset to another irreducible subset of same dimension. Thus  $S$  can be uniquely written as a disjoint union of  $G$ -invariant subsets. By taking union of closure of these generic points in each invariant subsets, we get the  $G$ -invariant components of  $X$ . Clearly, any nonempty intersection of  $U_i$  and  $U_j$  for  $i \neq j$  will give a proper  $G$ -invariant component, and this will contradict the minimality.  $\square$

We shall now look at some properties of  $G$ -sheaves (definition 1.1). Let the abelian category of all coherent  $G$ -sheaves be denoted by  $\mathcal{Coh}^G(X)$ . In Tohoku paper of Grothendieck [7] it was proved that  $\mathcal{QCoh}^G(X)$  has enough injectives. Therefore derived functors of various functors like  $\pi_*$ ,  $\pi^*$  and  $\otimes$  will always exist similar to non-equivariant case and for simplicity we shall use same notation. We shall denote the bounded derived category of coherent  $G$ -sheaves by  $\mathcal{D}^G(X)$ . Similar to the case of  $\mathcal{D}^b(X)$  we have a symmetric monoidal structure on  $\mathcal{D}^G(X)$  given by the (left) derived functor of tensor structure on  $\mathcal{Coh}^G(X)$ . Given an algebraic variety  $X$  with an action of a finite group  $G$  we have a natural morphism  $\pi : X \rightarrow Y$  which further gives a functor  $\pi_* : \mathcal{Coh}^G(X) \rightarrow \mathcal{Coh}(Y)$  and by taking  $G$ -invariant part of image we can define a functor  $\pi_*^G : \mathcal{Coh}^G(X) \rightarrow \mathcal{Coh}(Y)$  i.e.  $\pi_*^G(\mathcal{F}, \rho) = (\pi_*(\mathcal{F}, \rho))^G$  for all  $(\mathcal{F}, \rho) \in \mathcal{Coh}^G(X)$ . We have following result when  $G$  acts freely on  $X$ , see Mumford's book [12] for proof,

**Proposition 2.17.** *Let  $\pi : X \rightarrow Y$  be a natural morphism given by free action of the finite group  $G$  on  $X$ . The map  $\pi^* : \mathcal{Coh}(Y) \rightarrow \mathcal{Coh}^G(X)$  is an equivalence of abelian categories with the quasi-inverse  $\pi_*^G$ . Further locally free sheaves corresponds to locally free sheaves of the same rank.*

Now we can extend above equivalence to get a tensor equivalence  $\pi^*$  between tensor triangulated categories  $\mathcal{D}^b(Y)$  and  $\mathcal{D}^G(X)$ . In general these two categories are not equivalent. If we now take the case when  $G$  acts trivially on an algebraic variety  $X$  then we have the canonical decomposition of each objects of  $\mathcal{D}^G(X)$  similar to the case of finite dimensional representation of finite group which is a particular case of this by taking  $X$  to be a single point  $\text{Spec } k$ . So, more generally, we have following result,

**Proposition 2.18.** *Let  $X$  be an algebraic variety defined over  $k$  with a  $G$  action, and let  $H$  be a subgroup of  $G$  with the property that it acts trivially on  $X$ . Then any object  $(\mathcal{F}, \rho) \in \mathcal{D}^G(X)$  has the canonical decomposition as follows,*

$$(\mathcal{F}, \rho) = \oplus_{\lambda} W_{\lambda} \otimes (\mathcal{F}, \rho)_{\lambda}$$



where  $(\mathcal{F}, \rho)_\lambda = (W_\lambda^* \otimes (\mathcal{F}, \rho))^H$  and  $W_\lambda$  is a finite dimensional representation of the subgroup  $H$  with the natural action of the group  $G/H$  on  $(\mathcal{F}, \rho)_\lambda$ .

*Proof.* In any  $k$ -linear category we can identify  $W \otimes \mathcal{G}$ , where  $W$  is a finite dimensional vector space, with finite direct sum of object  $\mathcal{G}$ . Similarly we can define  $\text{Hom}(V, \mathcal{G}) := (V^* \otimes \mathcal{G})$  for any object  $\mathcal{G}$  and also we get the natural evaluation map  $ev : V \otimes \text{Hom}(V \otimes \mathcal{G}) \rightarrow \mathcal{G}$ . Moreover if  $V$  and  $\mathcal{G}$  have  $G$ -action then  $ev$  is  $G$ -equivariant map. In our situation using *déviissage* it is enough to study this  $ev$  map for pure sheaf, say  $(\mathcal{F}, \rho)$ . Now considering the induced action of  $H$  on  $(\mathcal{F}, \rho)$  we get a map,

$$ev : \oplus_\lambda W_\lambda \otimes (\mathcal{F}, \rho)_\lambda \rightarrow (\mathcal{F}, \rho).$$

We shall prove that the map  $ev$  is an isomorphism. Since this is a local question we can reduce to the case of  $A$ -module  $M$  with a  $G$ -action where  $A$  is  $k$ -algebra. Since the map  $ev$  is evidently a  $A$ -module morphism, it is enough to prove bijection of the map  $ev$  as a  $k$ -linear map. Restriction of this map is an isomorphism for any  $G$ -invariant finite dimensional vector subspace of  $M$ . This follows from the canonical decomposition of finite dimensional representation of  $G$ . Since any element is contained in a finite dimensional  $G$ -invariant subspace of  $M$  we get required bijection of the map  $ev$ . Since induced action of  $H$  is trivial on  $(\mathcal{F}, \rho)_\lambda$  we have action of quotient group  $G/H$ .  $\square$

Note that if  $G$  acts trivially on  $X$  then we can take  $H = G$  and as a particular case we shall get the canonical decomposition,

$$(\mathcal{F}, \rho) = \oplus_\lambda V_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where  $(\mathcal{F}, \rho)_\lambda = (V_\lambda^* \otimes (\mathcal{F}, \rho))^G$  and  $V_\lambda$  is a finite dimensional representation of the group  $G$ . Now we shall give a distinguished triangle for any complex of  $G$ -equivariant coherent sheaf  $\mathcal{F}$  over  $X$ . We have following result,

**Proposition 2.19.** *Let  $G$ ,  $k$  and  $X$  be as above.*

1. *Suppose  $U$  is any  $G$ -invariant open subset of  $X$  with induced action. If  $\pi$  denotes the projector  $\frac{1}{|G|} \sum_{g \in G} \rho_g^*$  on  $X$  where  $\rho_g$  is automorphism of  $X$  coming from the action of  $G$  then  $i_U^* \circ \pi = \pi \circ i_U^*$ . Here,  $\pi$  is also used to denote its restriction on open set  $U$ .*
2. *Suppose  $G$  acts faithfully on  $X$ . If  $\mathcal{F} \in \mathcal{D}^G(X)$  with  $\text{supph}(\mathcal{F}) = X$  then we have a distinguished triangle*

$$\pi^* \pi_*^G(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_1$$

*with  $\text{supph}(\mathcal{F}_1) \subsetneq \text{supph}(\mathcal{F})$ . Same is true if we have faithful action of  $G$  on  $\text{supph}(\mathcal{F}) \subsetneq X$ .*

*Proof of 1.* Since  $U$  is a  $G$ -invariant subset, each automorphism  $\rho_g$  of  $X$  induces an automorphism. For simplicity we use the same notation  $\rho_g$ . Now assertion immediately follows from the following commutative square,

$$\begin{array}{ccc} U & \xrightarrow{i_U} & X \\ \downarrow \rho_g & & \downarrow \rho_g \\ U & \xrightarrow{i_U} & X \end{array}$$

and additivity.

*Proof of 2.* Since  $G$  acts faithfully on  $X$  we can use proposition 2.16 to get an open subset  $U \subseteq X$  with free action of group  $G$ . We shall use induction on amplitude length,  $\text{ampl}(\mathcal{F})$ . When  $\text{ampl}(\mathcal{F}) = 1$  then  $\mathcal{F}$  is a shift of a coherent sheaf so enough to prove for coherent sheaf. Now using the fact that  $\text{supp}(\mathcal{F}) = X$  we have  $i_U^*(\mathcal{F}) \neq 0$ . There is a natural morphism coming from adjunction and inclusion of  $G$ -invariant part, say  $\eta : \pi^* \pi_*^G(\mathcal{F}) \rightarrow \mathcal{F}$ . Using flat base change and part 1 of 2.19 we get an isomorphism  $i_U^* \pi^* \pi_*^G(\mathcal{F}) \simeq \pi^* \pi_*^G(i_U^* \mathcal{F})$ . Now this will give an isomorphism, as  $G$  act freely on  $U$ , i.e.  $i_U^*(\eta) : i_U^* \pi^* \pi_*^G(\mathcal{F}) \rightarrow i_U^* \mathcal{F}$  is an isomorphism. Hence cone of the map  $\eta$  will have support outside an open set  $U$ . This completes the first step of induction.

Now assume the for all  $\mathcal{F}$  with  $\text{ampl}(\mathcal{G}) \leq (n-1)$  we have such a distinguished triangle. Now consider  $\mathcal{F}$  with  $\text{ampl}(\mathcal{F}) = n$  with highest cohomology in degree  $n$ . We have usual truncation distinguished triangle  $\tau^{\leq(n-1)}(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \mathcal{H}^n(\mathcal{F})[-n]$ . Using exactness of  $i_U^*$  and argument similar to first step of induction we have a following commutative diagram (we have used same notation  $\eta$  for different sheaves),

$$\begin{array}{ccccc} i_U^* \pi^* \pi_*^G \tau^{\leq(n-1)}(\mathcal{F}) & \longrightarrow & i_U^* \pi^* \pi_*^G \mathcal{F} & \longrightarrow & i_U^* \pi^* \pi_*^G \mathcal{H}^n(\mathcal{F})[-n] \\ \downarrow i_U^*(\eta) & & \downarrow i_U^*(\eta) & & \downarrow i_U^*(\eta) \\ i_U^* \tau^{\leq(n-1)}(\mathcal{F}) & \longrightarrow & i_U^* \mathcal{F} & \longrightarrow & i_U^* \mathcal{H}^n(\mathcal{F})[-n] \end{array}$$

Since both the extreme vertical arrows are isomorphism using induction hypothesis, we have isomorphism of the middle  $i_U^*(\eta)$ . Therefore cone of the map  $\eta$  will have proper support.  $\square$

### 3 Example : Derived category of equivariant sheaves

In this section we shall compute Balmer's triangular spectrum for some particular examples. This computation of triangular spectrum also motivates the need for some finer geometric structures attached to a given tensor triangulated category.

#### 3.1 Equivariant sheaves

In this example we shall compute Balmer's triangular spectrum for equivariant sheaves over some quasi-projective varieties with  $G$ -action. We shall always consider the varieties over some fixed field  $k$ . We shall first do some particular cases before going to general case. The general case will be done in the next subsection.

##### Case 1: $X$ is a point

Let  $G$  be a finite group and  $k$  be any field of char 0. As usual  $\mathcal{R}\text{ep}(G)$  is the category of all finite dimensional  $k$  linear representation of a group  $G$ . We can define a strict symmetric monoidal structure on this category using the usual tensor product of representations i.e. if  $V_1$  and  $V_2$  are two representations of  $G$

then  $V_1 \otimes V_2$  is the tensor product as  $k$  vector spaces with diagonal action. We shall denote the bounded derived category of abelian category  $\mathcal{R}\mathfrak{ep}(G)$  (resp.  $\mathcal{R}\mathfrak{ep}(\{0\})$ ), for the trivial group  $\{0\}$  as  $\mathcal{D}_{k[G]}$  (resp.  $\mathcal{D}_k$ ). We can extend the above tensor product of representations to get a symmetric tensor triangulated structure on  $\mathcal{D}_{k[G]}$ .

**Proposition 3.1.**  $\text{Spec}(\mathcal{D}_{k[G]}) \cong \text{Spec}(\mathcal{D}_k) \cong \text{Spec}(k)$ .

*Proof.* Since  $\mathcal{R}\mathfrak{ep}(\{0\})$  is a semisimple abelian category with  $k$  as its unit it is easy to see that  $\text{Spec}(\mathcal{D}_k) \cong \text{Spec}(k)$  as a variety. Therefore it is enough to prove the first isomorphism. The unit object of  $\mathcal{D}_{k[G]}$  is  $k$  with endomorphism ring isomorphic to  $k$  so it remains to say that the trivial ideal, i.e. ideal with only zero object, is the only prime ideal. This follows from the following lemma.

**Lemma 3.2.** *Any representation of a finite group contains the trivial representation as a direct summand of some tensor power.*

*Proof.* Let  $V$  be a finite dimensional representation of a finite group  $G$ . Since  $\text{char}(k)$  is 0 we have an graded vector space isomorphism of the symmetric algebra with the symmetric tensors contained in tensor algebra  $T(V)$ , i.e. subspace of  $T^n(V)$  fixed by natural action of symmetric group  $S_n$  for all  $n$ .

In fact, this isomorphism is given by a section of the natural quotient map from  $T(V)$  to  $S(V)$  i.e.

$$v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Now if we take the image of a nonzero element  $\prod_{g \in G} g.v \in S^{|G|}(V)$  under this isomorphism then we shall get a nonzero fixed vector of  $|G|$ -symmetric tensors and hence it will give the trivial representation as a direct summand of  $V^{\otimes |G|}$  using the semisimplicity of  $\mathcal{R}\mathfrak{ep}(G)$ .  $\square$

Using the above lemma and thickness it is easy to see that any non trivial ideal is full.  $\square$

But for the sake of generalisation we shall give another proof of the first isomorphism.

Consider the two exact tensor functors  $F : \mathcal{D}_{k[G]} \rightarrow \mathcal{D}_k$  and  $G : \mathcal{D}_k \rightarrow \mathcal{D}_{k[G]}$  where  $F$  is the forgetful functor and  $G$  comes from the augmentation map of the group algebra  $k[G]$  i.e. sending each complex of vector space to a complex of  $k[G]$  module with the trivial action of a group  $G$ . Note that  $F \circ G = \text{Id}$  and hence  $\text{Spec}(G) \circ \text{Spec}(F) = \text{Id}$ . We now prove the following lemma.

**Lemma 3.3.**  $\text{Spec}(F) \circ \text{Spec}(G) = \text{Id}$ .

*Proof.* Let  $\mathcal{P} \in \text{Spec}(\mathcal{D}_{k[G]})$  be a prime ideal. We want to prove that  $(G \circ F)^{-1}(\mathcal{P}) = \mathcal{P}$ . If  $V \in \mathcal{M}\mathfrak{od}(k[G])$  is any  $k[G]$ -module, then we have the canonical decomposition,

$$V = \bigoplus_{\lambda} V_{\lambda} \otimes (V_{\lambda}^* \otimes V)^G$$

where  $V_{\lambda}$  is an irreducible representation of a group  $G$ . Further  $(V_{\lambda}^* \otimes V)^G$  is a direct summand of  $(V_{\lambda}^* \otimes V)$  as is seen using the projector  $\frac{1}{|G|} \sum_{g \in G} \rho_g$  where

$\rho_g$  comes from the action of a group  $G$  on  $(V_\lambda^* \otimes V)$ . Since any complex in  $\mathcal{D}_{k[G]}$  is isomorphic to the direct sum of translates of the cohomology of that complex, to prove above assertion its enough to prove that  $(G \circ F)^{-1}(\mathcal{P} \cap \mathcal{Mod}(k[G])) = \mathcal{P} \cap \mathcal{Mod}(k[G])$ . Observe that,

$$\begin{aligned}
V \in (\mathcal{P} \cap \mathcal{Mod}(k[G])) &\Leftrightarrow (V_\lambda \otimes V)^G \in (\mathcal{P} \cap \mathcal{Mod}(k[G])) \\
&\quad \text{using thickness and additivity} \\
&\Leftrightarrow (V_\lambda \otimes V)^G \in (G \circ F)^{-1}(\mathcal{P} \cap \mathcal{Mod}(k[G])) \\
&\quad \text{Since } (G \circ F)(W) = W \text{ if } G \text{ acts trivially on } W \\
&\Leftrightarrow V \in (G \circ F)^{-1}(\mathcal{P} \cap \mathcal{Mod}(k[G])) \\
&\quad \text{using thickness and additivity.}
\end{aligned}$$

Therefore above observation completes the proof of lemma.  $\square$

Hence using the above lemma we have another proof of the first isomorphism.

**Case 2:  $X$  smooth variety with a trivial  $G$  action.**

In this case, we shall extend the above example. Let  $X$  be a smooth variety considered as a space with the trivial action of a finite group  $G$ . Recall the definitions and some properties of a  $G$ -sheaves from the preliminary section 2. Let  $\mathcal{Coh}(X)$  (resp.  $\mathcal{Coh}^G(X)$ ) be the abelian category of all coherent sheaves (resp. coherent  $G$ -sheaves) over  $X$ . We have two functors  $F$  and  $G$  similar to the previous example defined as follows,

$$\begin{aligned}
F : \mathcal{Coh}^G(X) &\rightarrow \mathcal{Coh}(X) & \& \quad G : \mathcal{Coh}(X) \rightarrow \mathcal{Coh}^G(X) \\
(\mathcal{F}, \rho) &\mapsto \mathcal{F} & \mathcal{F} &\mapsto (\mathcal{F}, id)
\end{aligned}$$

Note that the functor  $F$  (respectively  $G$ ) is a faithful (respectively fully faithful) exact functor. Thus we get two exact derived functors of the above two functors,  $F : \mathcal{D}^G(X) \rightarrow \mathcal{D}^b(X)$  and  $G : \mathcal{D}^b(X) \rightarrow \mathcal{D}^G(X)$  which by abuse of notation are denoted by the same symbols.

Recall that  $\mathcal{D}^G(X)$  and  $\mathcal{D}^b(X)$  are a tensor triangulated categories which makes the functors  $F$  and  $G$  unital tensor functors and hence using the functorial property of “Spec” we shall get two morphisms  $\text{Spec}(F) : \text{Spec}(\mathcal{D}^b(X)) \rightarrow \text{Spec}(\mathcal{D}^G(X))$  and  $\text{Spec}(G) : \text{Spec}(\mathcal{D}^G(X)) \rightarrow \text{Spec}(\mathcal{D}^b(X))$ . Now we have following result,

**Proposition 3.4.**  $\text{Spec}(\mathcal{D}^G(X)) \cong \text{Spec}(\mathcal{D}^b(X)) \cong X$ .

*Proof.* Here, the second isomorphism was proved by Balmer [2] which enables him to reconstruct the variety from its associated tensor triangulated category of coherent sheaves. We shall use the idea of previous example to prove the first isomorphism.

Since  $F \circ G = Id$ , functoriality of the “Spec” will give  $\text{Spec}(G) \circ \text{Spec}(F) = Id$ . Now it remains to prove that  $\text{Spec}(F) \circ \text{Spec}(G) = Id$ . Note that every object  $(\mathcal{F}, \rho) \in \mathcal{D}^G(X)$  has the canonical decomposition as follows,

$$(\mathcal{F}, \rho) = \bigoplus_{\lambda} V_{\lambda} \otimes (\mathcal{F}, \rho)_{\lambda}$$

where  $(\mathcal{F}, \rho)_\lambda = (V_\lambda^* \otimes (\mathcal{F}, \rho))^G$  and  $V_\lambda$  is a finite dimensional irreducible representation of the group  $G$ , see section 2 for proof. Also note that  $(\mathcal{F}, \rho)_\lambda$  is an ordinary sheaf with the trivial action of a group  $G$  and also using similar projector as above, i.e.  $\frac{1}{|G|} \sum_{g \in G} \rho_g$ , we can prove that  $(\mathcal{F}, \rho)_\lambda$  is a direct summand of the sheaf  $(V_\lambda^* \otimes (\mathcal{F}, \rho))$ . Now we use the following lemma.

**Lemma 3.5.**  $\text{Spec}(F) \circ \text{Spec}(G) = \text{Id}$ .

*Proof.* Let  $\mathcal{P} \in \text{Spec}(\mathcal{D}^G(X))$  be a prime ideal. We want to prove that  $(G \circ F)^{-1}(\mathcal{P}) = \mathcal{P}$ . Now using the canonical decomposition of each objects of the triangulated category  $\mathcal{D}^G(X)$ , we have,

$$\begin{aligned} (\mathcal{F}, \rho) \in \mathcal{P} &\Leftrightarrow (\mathcal{F}, \rho)_\lambda \in \mathcal{P} \text{ using thickness, additivity and projector} \\ &\Leftrightarrow (\mathcal{F}, \rho)_\lambda \in (G \circ F)^{-1}(\mathcal{P}) \\ &\quad \text{Since } (G \circ F)(\mathcal{F}, id) = (\mathcal{F}, id) \text{ if } G \text{ acts trivially i.e. } \rho = id \\ &\Leftrightarrow (\mathcal{F}, \rho) \in (G \circ F)^{-1}(\mathcal{P}) \\ &\quad \text{using thickness, additivity and projector.} \end{aligned}$$

Hence the above observation completes the proof of lemma.  $\square$

Now, using the above lemma, it follows that  $\text{Spec}(F)$  is an isomorphism between  $\text{Spec}(\mathcal{D}^G(X))$  and  $\text{Spec}(\mathcal{D}^b(X))$ .  $\square$

*Remark 3.6.* 1. In case 1 the second proof also works for fields with characteristic co-prime to order of the group  $G$ .

2. The proof for the case of trivial action on smooth varieties doesn't need assumption of quasi-projectivity on the variety  $X$  which is used later for the general case for the existence of group quotients.

### Case 3: $G$ acts freely on a smooth variety $X$

Now we shall consider the case where a finite group  $G$  acts freely on  $X$ . We refer to section 2 for the definition. Recall that we have a canonical map  $\pi : X \rightarrow Y := X/G$  which is a  $G$ -equivariant map with the trivial action of  $G$  on  $Y$ . Now we can also define two functors associated with  $\pi$ :  $\pi^* : \mathcal{Coh}(Y) \rightarrow \mathcal{Coh}^G(X)$  and  $\pi_*^G : \mathcal{Coh}^G(X) \rightarrow \mathcal{Coh}(Y)$  where  $\pi_*^G = G$ -equivariant part of  $\pi_*$ . We had also seen in 2 that  $\pi^*$  is a tensor functor in general; and when  $G$  acts freely it is also an equivalence of categories with  $\pi_*^G$  as its quasi-inverse. Hence we shall get an equivalence of the tensor triangulated categories  $\mathcal{D}^b(Y)$  and  $\mathcal{D}^G(X)$ . Since an equivalence gives an isomorphism of "Spec", (cf. 2), therefore we get an isomorphism  $\text{Spec}(\pi^*) : \text{Spec}(\mathcal{D}^G(X)) \rightarrow \text{Spec}(\mathcal{D}^b(Y))$  with its inverse given by  $\text{Spec}(\pi_*^G)$ . In fact using case 2 and this argument, we can give slightly more general statement as follows.

**Corollary 3.7.** *Suppose finite group  $G$  acts freely on a quasi-projective variety  $X$  modulo some normal subgroup  $H$ . In other words, the subgroup  $H$  acts trivially, and the induced action of the quotient group  $G/H$  is free. Then*

$$\text{Spec}(\mathcal{D}^G(X)) \cong \text{Spec}(\mathcal{D}^b(Y)) \cong Y$$

where  $Y := X/G$  as before.

*Proof.* As mentioned above, the proof goes in similar lines as in case 2, using a more general canonical decomposition of objects of  $\mathcal{D}^G(X)$ :

$$(\mathcal{F}, \rho) = \bigoplus_{\lambda} W_{\lambda} \otimes (\mathcal{F}, \rho)_{\lambda}$$

where  $(\mathcal{F}, \rho)_{\lambda} = (W_{\lambda}^* \otimes (\mathcal{F}, \rho))^H$ ,  $W_{\lambda}$  is a finite dimensional irreducible representation of the group  $H$ , and the group  $G/H$  acts naturally on  $(\mathcal{F}, \rho)_{\lambda}$ . See section 2 for the proof.  $\square$

Finally we tackle the general case. Since the proof is a bit long involving some steps we devote a full subsection to it.

### 3.2 Case 4: The general case

Finally in this case we shall consider the more general situation of a finite group  $G$  acting on a smooth quasi-projective variety  $X$  and we further assume that the group  $G$  acts faithfully. Define  $\pi : X \rightarrow Y := X/G$  as above an  $G$ -equivariant map when considered with the trivial action of a group  $G$  on  $Y$ . Note that for a finite group the quotient space always exists [2]. We have a following main result,

**Proposition 3.8.**  $\text{Spec}(\mathcal{D}^G(X)) \cong \text{Spec}(\mathcal{D}^{per}(Y)) \cong Y$ .

Here again as before the second isomorphism is a particular case of the more general reconstruction result of Balmer [1] [2]. Hence we shall just prove the first isomorphism. We know there are two exact functors  $\pi^* : \mathcal{D}^{per}(Y) \rightarrow \mathcal{D}^G(X)$  and  $\pi_* : \mathcal{D}^G(X) \rightarrow \mathcal{D}^{per}(Y)$ . We also know that the map  $\pi^*$  is an unital tensor functor and hence it will give the map  $\text{Spec}(\pi^*) : \text{Spec}(\mathcal{D}^G(X)) \rightarrow \text{Spec}(\mathcal{D}^{per}(Y))$ . Note that  $\pi_*$  need not be a tensor functor. We shall prove that  $\text{Spec}(\pi^*)$  is a closed bijection and induces an isomorphism for the structure sheaves. To simplify the proof we will break it in several steps.

**Step 1:  $\text{Spec}(\pi^*)$  is onto.**

Suppose  $\mathfrak{q} \in \text{Spec}(\mathcal{D}^{per}(Y))$  is a prime ideal then we want to construct an prime ideal  $\mathfrak{p}$  in  $\text{Spec}(\mathcal{D}^G(X))$  such that  $\mathfrak{q} = (\pi^*)^{-1}(\mathfrak{p})$ . Recall that  $\langle \pi^*(\mathfrak{q}) \rangle$  denotes the thick tensor ideal generated by the image of  $\mathfrak{q}$  via functor  $\pi^*$  in a tensor triangulated category  $\mathcal{D}^G(X)$ . We have a following lemma which uses the explicit description of thick tensor ideal  $\langle \pi^*(\mathfrak{q}) \rangle$ .

**Lemma 3.9.**  $\pi_*(\langle \pi^*(\mathfrak{q}) \rangle) \subseteq \mathfrak{q}$ .

*Proof.* To prove this lemma, we use lemma2.7 i.e.

$$\langle \pi^*(\mathfrak{q}) \rangle = \bigcup_{n \geq 0} \langle \pi^*(\mathfrak{q}) \rangle^n$$

where  $\langle \pi^*(\mathfrak{q}) \rangle^n$  constructed inductively by taking  $\langle \pi^*(\mathfrak{q}) \rangle^0$  as the summands of tensor ideal generated by  $\pi^*(\mathfrak{q})$  and  $\langle \pi^*(\mathfrak{q}) \rangle^n$  to be the thick tensor ideal containing cone of morphism between any two objects of  $\langle \pi^*(\mathfrak{q}) \rangle^{(n-1)}$  and  $\langle \pi^*(\mathfrak{q}) \rangle^0$ . Here cone of a morphism refers to the third object of any distinguished triangle having this morphism as a base or equivalently we can use  $\diamond$  operation. The above equality follows from the lemma2.7 proved earlier.

We shall use induction on  $n$  in the above explicit description. For  $n = 0$ , given  $\mathcal{F} \in \mathfrak{q}$ ,

$$\pi_*(\pi^*(\mathcal{F}) \otimes \mathcal{G}) = \mathcal{F} \otimes \pi_*(\mathcal{G}) \in \mathfrak{q},$$

and hence  $\pi_*(\langle \pi^*(\mathfrak{q}) \rangle^0) \subseteq \mathfrak{q}$  using thickness of  $\mathfrak{q}$ .

Using induction suppose we know that  $\pi_*(\langle \pi^*(\mathfrak{q}) \rangle^{(n-1)}) \subseteq \mathfrak{q}$ . Since  $\pi_*$  is an exact functor, it follows that the image under  $\pi_*$  of a cone of any morphism is a cone of  $\pi_*$  of the morphism. Hence using the triangulated ideal property and thickness of  $\mathfrak{q}$  it follows that  $\pi_*(\langle \pi^*(\mathfrak{q}) \rangle^n) \subseteq \mathfrak{q}$ . Therefore we have  $\pi_*(\langle \pi^*(\mathfrak{q}) \rangle) = \pi_*(\bigcup_{n \geq 0} \langle \pi^*(\mathfrak{q}) \rangle^n) \subseteq \mathfrak{q}$ .  $\square$

**Lemma 3.10.**  $\pi^*(\mathcal{D}^{per}(Y) \setminus \mathfrak{q}) \cap \langle \pi^*(\mathfrak{q}) \rangle = \emptyset$ .

*Proof.* To prove this by contradiction, suppose that there exists an object  $\mathcal{G} \in (\mathcal{D}^{per}(Y) \setminus \mathfrak{q})$  such that  $\pi^*(\mathcal{G}) \in \langle \pi^*(\mathfrak{q}) \rangle$ . Then using the above lemma  $\pi_*(\pi^*(\mathcal{G})) \in \mathfrak{q}$ . On the other hand, the projection formula implies  $\pi_*(\pi^*(\mathcal{G})) = \mathcal{G} \otimes \pi_*(\mathcal{O}_X)$ , which we saw is in  $\mathfrak{q}$ .

Using the primality of  $\mathfrak{q}$  it follows that  $\pi_*(\mathcal{O}_X) \in \mathfrak{q}$ . Now  $(\pi_*(\mathcal{O}_X))^G = \mathcal{O}_Y$  is a direct summand of  $\pi_*(\mathcal{O}_X)$  by the canonical decomposition of a  $G$ -sheaves on  $Y$ . Hence  $\mathcal{O}_Y$  is an object of  $\mathfrak{q}$ ; which is absurd.  $\square$

To complete Step 1, we apply Balmer's result 2.10 to get an prime ideal  $\mathfrak{p}$ , such that  $\pi^*(\mathcal{D}^{per}(Y) \setminus \mathfrak{q}) \cap \mathfrak{p} = \emptyset$  and  $\langle \pi^*(\mathfrak{q}) \rangle \subseteq \mathfrak{p}$ . Hence we shall get  $\mathfrak{q} = (\pi^*)^{-1}(\mathfrak{p})$  which proves the surjectivity of the map  $\text{Spec}(\pi^*)$ .

## Step 2: Injectivity of $\text{Spec}(\pi^*)$

First we shall give proof for the case of a *smooth projective curve* as it is simpler than the general case. We have a following basic result for the case of a smooth projective curve which we shall use in the proof.

**Proposition 3.11.** 1. Any object of  $\mathcal{D}^b(\mathcal{A})$ , for a hereditary abelian category  $\mathcal{A}$ , is noncanonically isomorphic to the direct sum of its cohomologies with shifts. In particular, this is true for  $\mathcal{A} = \text{Coh}(X)$  where  $X$  is a smooth projective curve.

2. Every coherent sheaf over smooth projective curve  $X$  is a direct sum of a coherent skyscraper sheaves and a locally free coherent sheaves.

Using above result we prove the following proposition.

**Proposition 3.12.** The map  $\text{Spec}(\pi^*) : \text{Spec}(\mathcal{D}^G(X)) \rightarrow \text{Spec}(\mathcal{D}^b(Y))$  is an injective map between smooth projective curves  $X$  and  $Y$ .

*Proof.* Suppose not, let  $\mathfrak{p}_1, \mathfrak{p}_2$  be two distinct points of  $\text{Spec}(\mathcal{D}^G(X))$  mapping to the same point  $\mathfrak{q}_y$  where  $y$  is given by the identification of  $\text{Spec}(\mathcal{D}^b(Y))$  with  $Y$ . Let  $\mathcal{F}$  be an element of  $\mathfrak{p}_1$  and using the above proposition (3.11) we can assume that it is a pure sheaf. We have the following lemma which gives a restriction on the homological support of such elements.

**Lemma 3.13.**  $\text{supp}(\mathcal{F}) \subseteq (X \setminus \pi^{-1}(y))$ .



First let us complete the proof of the proposition assuming this lemma. From the lemma it follows that  $\text{supp}(\mathcal{F})$  is a proper subset of  $X$  with a  $G$ -action. Therefore  $\text{supp}(\mathcal{F})$  is a finite set of points and using thickness further we can assume that it is a single orbit. Suppose  $H$  is a stabiliser of this orbit. Then  $G/H$  will act freely on  $\text{supp}(\mathcal{F})$ . Now we have the decomposition,

$$\mathcal{F} = \oplus_{\lambda} W_{\lambda} \otimes \mathcal{F}_{\lambda} \simeq \oplus_{\lambda} W_{\lambda} \otimes \pi^* \pi_*^{G/H}(\mathcal{F}_{\lambda})$$

where  $W_{\lambda}$  is an irreducible representation of  $H$ . Therefore  $\mathcal{F} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ , since using a projector  $\mathcal{F}_{\lambda} = (W_{\lambda}^* \otimes \mathcal{F})^G \simeq \pi^* \pi_*^{G/H}(\mathcal{F}_{\lambda}) \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ , and hence  $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ . Using similar arguments we can prove  $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ . This is a contradiction as  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  are distinct points.

This proves the proposition assuming the lemma. Next we prove the lemma.  $\square$

*Proof of lemma.* We prove it by contradiction. Assume  $\text{supp}(\mathcal{F}) \cap \pi^{-1}(y) \neq \emptyset$ . If  $y$  is a closed point then we can assume that  $\text{supp}(\mathcal{F}) = \pi^{-1}(y)$  since we can always tensor with the object  $\mathcal{O}_{\pi^{-1}(y)}$  which will give an object of  $\mathfrak{p}_1$ . And if  $H$  is a stabiliser of this finite  $G$  set then we shall have the usual decomposition  $\mathcal{F} = \oplus_{\lambda} W_{\lambda} \otimes \pi^* \pi_*^{G/H}(\mathcal{F}_{\lambda})$ . Hence we shall get an object,  $\pi_*^{G/H}(\mathcal{F}_{\lambda})$ , of  $\mathfrak{q}_y$  supported on  $y$  which is a contradiction.

Similarly, if  $y$  is a generic point of  $X$  then using the above proposition 3.11 we can assume that  $\mathcal{F}$  is a  $G$ -equivariant vector bundle. Now using a short exact sequence, inspired from short exact sequence (4.7) from paper[15],  $0 \rightarrow \pi^* \pi_*^G(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$  with  $\mathcal{F}'$  supported on a points, we can prove  $\mathcal{F}' \in \mathfrak{p}_1$  and hence  $\pi^* \pi_*^G(\mathcal{F}) \in \mathfrak{p}_1$ . Now using our assumption  $\pi_*^G(\mathcal{F}) \in \mathfrak{q}_y$  which is a contradiction as  $\pi_*^G(\mathcal{F})$  is a vector bundle.  $\square$

This finishes the case of curves. For the general case we need the following proposition.

**Proposition 3.14.** *The map  $\text{Spec}(\pi^*) : \text{Spec}(\mathcal{D}^G(X)) \rightarrow \text{Spec}(\mathcal{D}^{per}(Y))$  is an injective map where  $X$  is a smooth quasi-projective varieties of dimension  $n$ .*

*Proof.* Suppose not, let  $\mathfrak{p}_1, \mathfrak{p}_2$  be two distinct points of  $\text{Spec}(\mathcal{D}^G(X))$  which maps to the same point  $\mathfrak{q}_y$  i.e.  $(\pi^*)^{-1}(\mathfrak{p}_1) = (\pi^*)^{-1}(\mathfrak{p}_2) = \mathfrak{q}_y$ . Let  $\mathcal{F} \in \mathfrak{p}_1$  be an complex of  $G$ -equivariant sheaves. We need following lemma.

**Lemma 3.15.** *1. There exists a tower of distinguished triangles for each element  $\mathcal{F}$ ,*

$$\begin{array}{ccccccc} \mathcal{F} = \mathcal{F}_0 & \xrightarrow{\quad} & \mathcal{F}_1 & \cdots & \mathcal{F}_{m-1} & \xrightarrow{\quad} & \mathcal{F}_m \\ & \nwarrow & \nearrow & & \nwarrow & \nearrow & \searrow \\ & & \mathcal{G}_1 & \cdots & \mathcal{G}_{m-1} & & \mathcal{G}_m \end{array}$$

where  $\mathcal{G}_i = \bigoplus_{\lambda_i} W_{\lambda_i} \otimes \pi^* \pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$  with the sum being over the irreducible representations of the corresponding  $H_i$ 's,  $\text{supph}(\mathcal{F}_m) \subsetneq \cdots \subsetneq \text{supph}(\mathcal{F})$  and  $\text{supph}(\pi_*^{G/H_j}(\mathcal{F}_{\lambda_j})) \subseteq \pi(\text{supph}(\mathcal{F}_{\lambda_j})) = \text{supph}(\pi_*(\mathcal{F}_{\lambda_j}))$ .

2.  $\text{supph}(\mathcal{F}) \subseteq (X \setminus \pi^{-1}(y))$ .

Also note, we can prove that any  $\mathcal{F} \in \mathcal{D}^G(X)$  with the homological support contained in  $(X \setminus \pi^{-1}(y))$  will be an element of  $\mathfrak{p}_1$ .

We shall first complete the proof of the proposition assuming this lemma. By the lemma, the homological support of  $\mathcal{F}$  is a proper closed subset of  $X$  not containing  $\pi^{-1}(y)$ . Note that in above lemma similar to the case of  $\mathcal{F}$ 's we also have decreasing filtration of a homological supports for  $\mathcal{G}$ 's. If we start with  $\mathcal{F} \in \mathfrak{p}_1$  then using (2.) of the lemma we have  $\text{supph}(\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) \subseteq (Y \setminus y)$  and hence  $\mathcal{G}_1 \in \pi^*(\mathfrak{q}_y) \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ . Therefore by definition of the prime ideal  $\mathcal{F}_1 \in \mathfrak{p}_1$ . Again using lemma we have the restriction on homological support of  $\mathcal{F}_1$  which gives  $\mathcal{G}_2 \in \pi^*(\mathfrak{q}_y) \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ . Now continuing like this we can prove that  $\mathcal{G}_j \in \pi^*(\mathfrak{q}_y) \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$  for  $j = 1, \dots, m$ . In particular,  $\mathcal{F}_m \in \mathfrak{p}_2$  and hence  $\mathcal{F}_{m-1} \in \mathfrak{p}_2$ . Now continuing in the reverse direction we can prove that  $\mathcal{F} \in \mathfrak{p}_2$ . Thus  $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ . Similarly by symmetry we can prove that  $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ . This is a contradiction as  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  are distinct points.  $\square$

*Proof of the lemma. Proof of 1.* To prove the first part we consider the homological support of  $\mathcal{F}$  as  $G$ -subset of the  $G$ -set  $X$  and induct on dimension of it. If dimension is zero then it will be set of  $G$ -invariant points and we shall get the direct sums of skyscrapers on these points. If we have free action of  $G/H$  for some subgroup  $H$  then we shall have the canonical decomposition 2.18. This will prove first step of induction. Assume now for dimension of  $\text{supph}(\mathcal{F}) \leq (n-1)$  we have a tower as in statement of lemma. Now consider  $\mathcal{F}$  with dimension of  $\text{supph}(\mathcal{F}) = n$ . Here  $\text{supph}(\mathcal{F})$  is a union of  $G$ -invariant components and using the proposition 2.16 we shall get the open subsets  $U_i$  for  $i = 1, \dots, k$  and subgroups  $H_i$  for  $i = 1, \dots, k$ . As observed before these open sets are mutually disjoint and there is a free action of group  $G/H_i$  for  $i = 1, \dots, k$  on each  $U_i$  respectively. We can decompose  $i_U^*(\mathcal{F}) = \bigoplus_{i=1}^k i_{U_j}^*(\mathcal{F})$  for  $j = 1, \dots, k$ . Start with an open subset  $U_1$ . We have the decomposition 2.19 of  $i_{U_1}^*(\mathcal{F})$ ,

$$i_{U_1}^*(\mathcal{F}) = \bigoplus_{\lambda_1} W_{\lambda_1} \otimes \mathcal{F}_{\lambda_1}$$

where  $W_{\lambda_j}$  is an irreducible representation of subgroup  $H_1$ . In this decomposition, all the  $\mathcal{F}_{\lambda_1}$  are also  $G/H_1$ -sheaves over open subset  $U_1$ . Using adjunction and the inclusion we get a canonical isomorphism 2.19  $\eta : \pi^* \pi_*^{G/H_1}(\mathcal{F}_{\lambda_1}) \rightarrow \mathcal{F}_{\lambda_1}$  over open set  $U_1$ . This will give an isomorphism  $\eta : \pi^* \pi_*^{G/H_1}(i_{U_1}^* \mathcal{F}) \rightarrow i_{U_1}^* \mathcal{F}$  of objects over open set  $U_1$  by using additivity. Now this isomorphism gives an isomorphism  $i_{U_1}^*(\eta) : i_{U_1}^*(\pi^* \pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) \rightarrow i_{U_1}^*(\mathcal{F}_{\lambda_1})$ . This follows from flat base change and some functorial properties. For the diagram,

$$\begin{array}{ccc} U_1 & \xrightarrow{i_{U_1}} & X \\ \downarrow \pi & & \downarrow \pi \\ V_1 & \xrightarrow{i_{V_1}} & Y \end{array}$$

we have canonical isomorphisms,

$$\begin{aligned} i_{U_1}^*(\pi^* \pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) &\simeq \pi^* i_{V_1}^*(\pi_*(\mathcal{F}_{\lambda_1}))^{G/H_1} \simeq \pi^*(i_{V_1}^* \pi_*(\mathcal{F}_{\lambda_1}))^{G/H_1} \\ &\simeq \pi^*(\pi_* i_{U_1}^*(\mathcal{F}_{\lambda_1}))^{G/H_1} = \pi^* \pi_*^{G/H_1} i_{U_1}^*(\mathcal{F}_{\lambda_1}), \end{aligned}$$

which will imply that the map  $i_{U_1}^*(\eta)$  is an isomorphism. Therefore the cone of a map  $\eta$ , say  $\mathcal{F}_1$ , will have the property that  $i_{U_1}^*(\mathcal{F}_1) = 0$  and hence  $\text{supph}(\mathcal{F}_1) \subseteq (X \setminus U) \subsetneq \text{supph}(\mathcal{F})$ . And since  $\pi$  is an affine map then  $\pi_*$  will be exact and hence we can prove that  $\pi(\text{supph}(\mathcal{F}_{\lambda_1})) = \text{supph}(\pi_*(\mathcal{F}_{\lambda_1})) \supseteq \text{supph}(\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1}))$ . Now we can proceed similarly with  $\mathcal{F}_1$  which has less number of  $G$ -invariant components than  $\mathcal{F}$  and hence in finitely many steps (in less than  $k$  steps) dimension of homological support will drop. Hence we shall get  $\mathcal{F}_i$  and  $\mathcal{G}_i$  for  $i = 1, \dots, l$  with the stated restrictions on supports. The dimension of  $\text{supph}(\mathcal{F}_l) \leq (n-1)$  and hence using induction we are done.

*Proof of 2.* Suppose  $\text{supph}(\mathcal{F}) \cap \pi^{-1}(y) \neq \emptyset$  and hence we get  $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{\pi^{-1}(\bar{y})} \in \mathfrak{p}_1$ . Observe that  $\text{supph}(\mathcal{F}') = \pi^{-1}(\bar{y}) = \overline{\pi^{-1}(y)}$ . Now applying the same procedure as in 1., we shall get a distinguished triangle

$$\bigoplus_{\lambda} W_{\lambda} \otimes \pi^* \pi_*^{G/H}(\mathcal{F}'_{\lambda}) \rightarrow \mathcal{F}' \rightarrow \mathcal{F}'' \rightarrow$$

with  $\text{supph}(\mathcal{F}'') \subsetneq \text{supph}(\mathcal{F}')$  and hence again applying 1., we can prove that  $\mathcal{F}'' \in \langle \pi^*(\mathfrak{q}_y) \rangle \subseteq \mathfrak{p}_1$ . But this gives  $\pi^* \pi_*^{G/H}(\mathcal{F}'_{\lambda}) \in \mathfrak{p}_1$  with  $\text{supph}(\pi_*^{G/H}(\mathcal{F}'_{\lambda})) = \bar{y}$  which is a contradiction as  $\pi_*^{G/H}(\mathcal{F}'_{\lambda}) \notin \mathfrak{q}_y$ .  $\square$

### Step 3: $\text{Spec}(\pi^*)$ is closed and hence is a homeomorphism.

Here we need bijection of the above step to prove closedness of the map  $\text{Spec}(\pi^*)$ . We shall use the fact that  $W \otimes \mathcal{O}_X \notin \mathfrak{p}$  for any finite dimensional representation and any prime ideal  $\mathfrak{p}$ . Indeed this follows from the fact that every representation contains the trivial representation as a direct summand of some tensor power i.e.  $\mathcal{O}_X \subseteq (V_{\lambda} \otimes \mathcal{O}_X)^{\otimes |G|}$ . Since  $\text{supp}(a)$ ,  $a \in \mathcal{D}^G(X)$ , are the basic closed sets therefore it is enough to prove that image under the map  $\text{Spec}(\pi^*)$  are closed. Now to prove this we shall use the description given in lemma 3.15 for any object of  $\mathcal{D}^G(X)$ . Letting  $b_{\lambda_j} = \pi_*^{G/H_j}(a_{\lambda_j})$  for simplicity, we have the following lemma.

**Lemma 3.16.**  $\text{Spec}(\pi^*)(\text{supp}(a)) = \bigcup_j \bigcup_{\lambda_j} \text{supp}(b_{\lambda_j})$ .

*Proof.* Given  $a \in \mathfrak{p}$  we have  $b_{\lambda_j}$ 's as in lemma 3.15. Now,

$$\begin{aligned} a \in \mathfrak{p} &\Leftrightarrow W_{\lambda_j} \otimes \pi^*(b_{\lambda_j}) \in \mathfrak{p} \quad \forall j, \lambda_j \\ &\Leftrightarrow \pi^*(b_{\lambda_j}) \in \mathfrak{p}, \quad \text{since } W_{\lambda_j} \otimes \mathcal{O}_X \notin \mathfrak{p}. \end{aligned}$$

$$\text{Therefore} \quad a \notin \mathfrak{p} \Leftrightarrow \exists \lambda_j \text{ s.t. } \pi^*(b_{\lambda_j}) \notin \mathfrak{p}.$$

Let  $\mathfrak{p} \in \text{supp}(a)$  and hence by the definition  $a \notin \mathfrak{p}$ . Now using the above observation there exists a  $\lambda_j$  such that  $\pi^*(b_{\lambda_j}) \notin \mathfrak{p}$  i.e.  $b_{\lambda_j} \notin (\pi^*)^{-1}(\mathfrak{p}) = \text{Spec}(\pi^*)(\mathfrak{p})$  and hence  $\text{Spec}(\pi^*)(\mathfrak{p}) \in \text{supp}(b_{\lambda_j})$ . Therefore  $\text{Spec}(\pi^*)(\text{supp}(a)) \subseteq \bigcup_j \bigcup_{\lambda_j} \text{supp}(b_{\lambda_j})$ .

Conversely suppose  $\mathfrak{q} \in \bigcup_j \bigcup_{\lambda_j} \text{supp}(b_{\lambda_j})$  and hence  $\mathfrak{q} \in \text{supp}(b_{\lambda_j})$  for some  $\lambda_j$ . Therefore by definition  $b_{\lambda_j} \notin \mathfrak{q}$  but using the bijection of the map  $\text{Spec}(\pi^*)$  we have  $b_{\lambda_j} \notin (\pi^*)^{-1}(\mathfrak{p}) = \mathfrak{q}$  for some  $\mathfrak{p}$ . Now it follows that  $\pi^*(b_{\lambda_j}) \notin \mathfrak{p}$  and once again using the above observation we have  $a \notin \mathfrak{p}$  i.e.  $\mathfrak{p} \in \text{supp}(a)$ . Hence we have  $\bigcup_j \bigcup_{\lambda_j} \text{supp}(b_{\lambda_j}) \subseteq \text{Spec}(\pi^*)(\text{supp}(a))$ .  $\square$

Since union in right hand side of above lemma is finite it follows that the image of  $\text{supp}(a)$  under the map  $\text{Spec}(\pi^*)$  is closed for all  $a \in \mathcal{D}^G(X)$ . Hence the map  $\text{Spec}(\pi^*)$  is a closed map and therefore it is a homeomorphism.

**Step 4:  $\text{Spec}(\pi^*)$  is an isomorphism.**

In this step we shall prove that the above homeomorphism  $\text{spec}(\pi^*)$  is, in fact, an isomorphism. We begin by proving the following lemma which we shall use later.

**Lemma 3.17.** *There exist a natural transformation  $\eta : \pi^* \pi_*^G \rightarrow \text{Id}$  (resp.  $\mu : \text{Id} \rightarrow \pi_*^G \pi^*$ ) such that  $\eta(\mathcal{O}_X) = \text{id}$  (resp.  $\mu(\mathcal{O}_Y) = \text{id}$ ) where  $\pi^* \pi_*^G(\mathcal{O}_X) = \mathcal{O}_X$  (resp.  $\pi_*^G \pi^*(\mathcal{O}_Y) = \mathcal{O}_Y$ ).*

*Proof.* We shall prove the existence of  $\eta$ , as  $\mu$  can be found using similar arguments. Since the functor  $\pi^*$  is a left adjoint of the functor  $\pi_*$  we have a natural transformation  $\eta' : \pi^* \pi_* \rightarrow \text{Id}$  given by the adjunction property. We also have a natural transformation given by inclusion of  $G$ -invariant part of sheaves on  $Y$ , say  $I$ . Now composing with the functors  $\pi^*$  and  $\pi_*$  we get another natural transformation which composed with  $\eta'$  gives the  $\eta$  i.e.  $\eta := \eta' \circ (\pi^* \cdot I \cdot \pi_*)$ . Now to prove  $\eta(\mathcal{O}_X) = \text{Id}$  we can assume that  $X$  is an affine variety. Suppose  $\tilde{A}$  is a structure sheaf of  $X$ . As  $A$  is flat over  $A^G$  we can reduce to computing a map from  $\pi^* \pi_*^G(\tilde{A}) \rightarrow \tilde{A}$ , in place of its derived functors. Now clearly the multiplication map  $A \otimes ({}_B A)^G \rightarrow A$  is just inverse of the natural identification map of  $A$  with  $A \otimes ({}_B A)^G$ . Hence the map  $\eta(\mathcal{O}_X) : \tilde{A} \rightarrow \tilde{A}$  is an identity map. Similarly we can prove that  $\mu(\mathcal{O}_Y) = \text{Id}$ .  $\square$

Recall the definitions of structure sheaves and associated map of the sheaves given by the unital tensor functor of underlying tensor triangulated categories 2.2 i.e. given an unital functor  $\pi^* : \mathcal{D}^{\text{per}}(Y) \rightarrow \mathcal{D}^G(X)$  the morphism  $\text{Spec}(\pi^*)$  induces a map of the structure sheaves,  $\text{Spec}(\pi^*)^\# : \mathcal{O}_Y \rightarrow \mathcal{O}_X$ . We shall prove that this map is an isomorphism by observing that  $\text{Spec}(\pi^*)^\#(V)$  is an isomorphism for every open set  $V \subseteq \text{Spec}(\mathcal{D}^{\text{per}}(Y))$ . If we take  $U = \pi^{-1}(V)$ ,  $Z = Y \setminus V$  and  $Z' = X \setminus U$  then we have a functor  $\pi_V^* : \frac{\mathcal{D}^{\text{per}}(Y)}{\mathcal{D}_Z^{\text{per}}(Y)} \rightarrow \frac{\mathcal{D}^G(X)}{\mathcal{D}_{Z'}^G(X)}$  which will induce a map  $\text{Spec}(\pi^*)^\#(V) := \pi_V^* : \text{End}_{\frac{\mathcal{D}^{\text{per}}(Y)}{\mathcal{D}_Z^{\text{per}}(Y)}}(\mathcal{O}_Y) \rightarrow \text{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}_{Z'}^G(X)}}(\mathcal{O}_X)$ .

**Lemma 3.18.** *The map  $\pi_V^* : \text{End}_{\frac{\mathcal{D}^{\text{per}}(Y)}{\mathcal{D}_Z^{\text{per}}(Y)}}(\mathcal{O}_Y) \rightarrow \text{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}_{Z'}^G(X)}}(\mathcal{O}_X)$  is surjective.*

*Proof.* Suppose  $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y]$  is an element of  $\text{End}_{\frac{\mathcal{D}^{\text{per}}(Y)}{\mathcal{D}_Z^{\text{per}}(Y)}}(\mathcal{O}_Y)$  then the map  $\pi^*$  will send it to an element  $[\mathcal{O}_X \xleftarrow{\pi^*(s)} \pi^*(\mathcal{G}) \xrightarrow{\pi^*(a)} \mathcal{O}_X]$  of  $\text{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}_{Z'}^G(X)}}(\mathcal{O}_X)$ .

It is now enough to prove that this map is a bijection.

Let  $[\mathcal{O}_X \xleftarrow{t} \mathcal{F} \xrightarrow{b} \mathcal{O}_X] \in \text{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}_{Z'}^G(X)}}(\mathcal{O}_X)$  be a given element then us-

ing the functor  $\pi_*^G$  we shall get an element  $[\mathcal{O}_Y \xleftarrow{\pi_*^G(t)} \pi_*^G(\mathcal{F}) \xrightarrow{\pi_*^G(b)} \mathcal{O}_Y] \in \text{End}_{\frac{\mathcal{D}^{\text{per}}(Y)}{\mathcal{D}_Z^{\text{per}}(Y)}}(\mathcal{O}_Y)$  as  $\text{supp}(C(\pi_*^G(t))) \subseteq Z$  using the flat base change and the

canonical isomorphism,  $i_V^* \pi_*^G(\mathcal{F}) \simeq (i_V^* \pi_*(\mathcal{F}))^G \simeq \pi_*^G(i_U^* \mathcal{F}) \xrightarrow{i_U^*(t)} \pi_*^G(i_U^* \mathcal{O}_X) \simeq \mathcal{O}_Y$ . Now we want to prove that

$$[\mathcal{O}_X \xleftarrow{t} \mathcal{F} \xrightarrow{b} \mathcal{O}_X] = [\mathcal{O}_X \xleftarrow{\pi^* \pi_*^G(t)} \pi^* \pi_*^G(\mathcal{F}) \xrightarrow{\pi^* \pi_*^G(b)} \mathcal{O}_X].$$

Using the lemma 3.17, we have a natural map  $\eta(\mathcal{F}) : \pi^* \pi_*^G(\mathcal{F}) \rightarrow \mathcal{F}$ , so to prove the assertion it is now enough to check that  $t \circ \eta(\mathcal{F}) = \pi^* \pi_*^G(t)$ ,  $b \circ \eta(\mathcal{F}) =$

$\pi^* \pi_*^G(b)$  and the cone of  $\eta(\mathcal{F})$  is supported on  $Z'$  that is  $C(\eta(\mathcal{F})) \in \mathcal{D}_{Z'}^G(X)$ . Here the first two assertions follows from the following commutative diagrams which are a consequence of lemma 3.17.

$$\begin{array}{ccc} \pi^* \pi_*^G(\mathcal{F}) & \xrightarrow{\eta(\mathcal{F})} & \mathcal{F} \\ \pi^* \pi_*^G(t) \downarrow & & \downarrow t \\ \mathcal{O}_X & \xrightarrow{\eta(\mathcal{O}_X)} & \mathcal{O}_X \end{array} \quad \begin{array}{ccc} \pi^* \pi_*^G(\mathcal{F}) & \xrightarrow{\eta(\mathcal{F})} & \mathcal{F} \\ \pi^* \pi_*^G(b) \downarrow & & \downarrow b \\ \mathcal{O}_X & \xrightarrow{\eta(\mathcal{O}_X)} & \mathcal{O}_X \end{array}$$

Now the last assertion  $C(\eta(\mathcal{F})) \in \mathcal{D}_{Z'}^G(X)$  is equivalent to  $i_U^* C(\eta(\mathcal{F})) \simeq 0$  in  $\mathcal{D}^G(U)$  but as the functor  $i_U^*$  is exact this assertion is same as  $C(i_U^* \eta(\mathcal{F})) \simeq 0$ . Using a property of distinguished triangle it is enough to check that the map  $i_U^* \eta(\mathcal{F})$  is an isomorphism. And this follows from the following commutative diagram.

$$\begin{array}{ccc} i_U^* \pi^* \pi_*^G(\mathcal{F}) & \xrightarrow{i_U^* \eta(\mathcal{F})} & i_U^* \mathcal{F} \\ \downarrow \wr & & \parallel \\ \pi^* \pi_*^G(i_U^* \mathcal{F}) & \xrightarrow{\eta(i_U^* \mathcal{F})} & i_U^* \mathcal{F} \\ \pi^* \pi_*^G i_U^*(t) \downarrow \wr & & \downarrow i_U^*(t) \wr \\ \pi^* \pi_*^G(\mathcal{O}_U) & \xrightarrow{\eta(\mathcal{O}_U)} & \mathcal{O}_U \end{array}$$

In above diagram we had used the same notations  $\pi$  and  $\eta$  for its restriction on open subsets. Here the top left vertical isomorphism comes from the flat base change formula and using the following canonical isomorphism.

$$i_U^* \pi^* \pi_*^G(\mathcal{F}) \simeq \pi^* i_V^*(\pi_*(\mathcal{F}))^G \simeq \pi^*(i_V^* \pi_*(\mathcal{F}))^G \simeq \pi^*(\pi_* i_U^*(\mathcal{F}))^G = \pi^* \pi_*^G(i_U^* \mathcal{F}).$$

This proves that  $\pi_V^*$  is surjective.  $\square$

**Lemma 3.19.**  $\pi_V^*$  is injective.

*Proof.* Let  $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y] \in \text{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}_{Z'}^{per}(Y)}}(\mathcal{O}_Y)$  maps to zero in  $\text{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}_{Z'}^G(X)}}(\mathcal{O}_X)$

i.e.  $[\mathcal{O}_X \xleftarrow{\pi^*(s)} \pi^*(\mathcal{G}) \xrightarrow{\pi^*(a)} \mathcal{O}_X] = 0$  which is equivalent to the existence of  $\mathcal{F}$  and a map  $t : \mathcal{F} \rightarrow \pi^* \mathcal{G}$  with  $\text{supp}(C(t)) \subseteq Z'$  such that  $\pi^*(a) \circ t = 0$ . Now the map  $\pi_*^G(t) : \pi_*^G(\mathcal{F}) \rightarrow \pi_*^G \pi^*(\mathcal{G})$  gives  $\pi_*^G \pi^*(a) \circ \pi_*^G(t) = 0$  and as proved earlier we know that  $\text{supp}(C(\pi_*^G(t))) \subseteq Z$  whenever  $\text{supp}(C(t)) \subseteq Z'$ . Hence the element  $[\mathcal{O}_Y \xleftarrow{\pi_*^G \pi^*(s)} \pi_*^G \pi^*(\mathcal{G}) \xrightarrow{\pi_*^G \pi^*(a)} \mathcal{O}_Y] = 0$  in  $\text{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}_Z^{per}(Y)}}(\mathcal{O}_Y)$ . We

shall prove that  $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y] = [\mathcal{O}_Y \xleftarrow{\pi_*^G \pi^*(s)} \pi_*^G \pi^*(\mathcal{G}) \xrightarrow{\pi_*^G \pi^*(a)} \mathcal{O}_Y]$  as an elements of  $\text{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}_Z^{per}(Y)}}(\mathcal{O}_Y)$ . Now using lemma 3.17 we have a map

$\mu(\mathcal{G}) : \mathcal{G} \rightarrow \pi_*^G \pi^*(\mathcal{G})$  which gives the following commutative diagrams as before using lemma 3.17,

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\mu(\mathcal{G})} & \pi_*^G \pi^*(\mathcal{G}) \\ s \downarrow & & \downarrow \pi_*^G \pi^*(s) \\ \mathcal{O}_Y & \xrightarrow{\mu(\mathcal{O}_Y)} & \mathcal{O}_Y \end{array} \quad \begin{array}{ccc} \mathcal{G} & \xrightarrow{\mu(\mathcal{G})} & \pi_*^G \pi^*(\mathcal{G}) \\ a \downarrow & & \downarrow \pi_*^G \pi^*(a) \\ \mathcal{O}_Y & \xrightarrow{\mu(\mathcal{O}_Y)} & \mathcal{O}_Y \end{array}$$

Therefore it remains to prove that  $i_V^* C(\mu(\mathcal{G})) = 0$  but as before this is equivalent to proving  $C(i_V^* \mu(\mathcal{G})) = 0$  since the functor  $i_V^*$  is an exact functor. Again using the property of a distinguished triangle it is enough to prove that  $i_V^* \mu(\mathcal{G})$  is an isomorphism. This clearly follows from the following commutative diagrams,

$$\begin{array}{ccc}
i_V^* \mathcal{G} & \xrightarrow{i_V^* \mu(\mathcal{G})} & i_V^* \pi_*^G \pi^*(\mathcal{G}) \\
\parallel & & \downarrow \wr \\
i_V^* \mathcal{G} & \xrightarrow{\mu(i_V^* \mathcal{G})} & \pi_*^G \pi^*(i_V^* \mathcal{G}) \\
\downarrow i_V^*(s) \wr & & \downarrow \wr \pi_*^G \pi^* i_V^*(s) \\
\mathcal{O}_V & \xrightarrow{\mu(\mathcal{O}_V)} & \pi_*^G \pi^*(\mathcal{O}_V).
\end{array}$$

Here again as earlier the top right vertical isomorphism comes from the flat base change and the following sequence of natural isomorphisms.

$$i_V^* \pi_*^G \pi^*(\mathcal{G}) \simeq i_V^*(\pi_* \pi^* \mathcal{G})^G \simeq (i_V^* \pi_* \pi^* \mathcal{G})^G \simeq \pi_*^G i_U^* \pi^* \mathcal{G} \simeq \pi_*^G \pi^*(i_V^* \mathcal{G}).$$

This proves injectivity of the map  $\pi_V^*$ .  $\square$

From the above two lemmas it follows that  $\pi_V^*$  is an isomorphism and hence  $\text{Spec}(\pi^*)$  is an isomorphism of the varieties  $\text{Spec}(\mathcal{D}^{per}(Y))$  and  $\text{Spec}(\mathcal{D}^G(X))$ .

## 4 Example : Superschemes

In this section first we shall recall the basic definition of superscheme and some properties of it. We shall relate various notion for some superschemes with the usual scheme with certain diagram.

### 4.1 Superalgebra

An associative  $\mathbb{Z}/2\mathbb{Z}$ -grading ring is an associative ring  $R$  with direct sum decomposition  $R = R^0 \oplus R^1$  as an additive group with multiplication that preserves the grading i.e.  $R^i R^j \subseteq R^{i+j}$  for  $i, j \in \mathbb{Z}/2\mathbb{Z}$ . There exists a parity function which takes value in ring  $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$  for every homogeneous element of  $R$  i.e. if  $r \in R^i$  then parity denoted  $\bar{r} = i$ . Now we restrict to following important class of rings,

**Definition 4.1.** An associative  $\mathbb{Z}/2\mathbb{Z}$  graded ring with unity,  $R = R^0 \oplus R^1$  is called supercommutative if the supercommutator of ring  $R$  is zero i.e.  $[r_1, r_2] := r_1 r_2 - (-1)^{\bar{r}_1 \bar{r}_2} r_2 r_1 = 0$  for all  $r_1, r_2 \in R$ . Further ring is called  $k$ -superalgebra if  $R$  is supercommutative  $k$ -algebra with  $k \subseteq R^0$ .

As usual we can define an abelian category of left modules over any  $k$ -superalgebra  $R$ , say  $\mathcal{Mod}(R)$ . An object of this category is a  $\mathbb{Z}/2\mathbb{Z}$ -graded abelian group with left  $R$ -module structure which is compatible with grading i.e.  $R^i M^j \subseteq M^{i+j}$  for all  $i, j = 0, 1$ . Morphism between these objects is a grade preserving morphism compatible with action of  $R$ . Similarly there exists a parity function defined for each homogeneous element of module  $M$  and denoted by the same notation as before. We can define parity change functor

$\Pi : \mathcal{Mod}(R) \rightarrow \mathcal{Mod}(R); M \mapsto \Pi M$  with  $\mathbb{Z}/2\mathbb{Z}$  grading given by  $(\Pi M)^0 = M^1$  and  $(\Pi M)^1 = M^0$ . There exists a two exact faithful functors from  $\mathcal{Mod}(R)$  as follows,

$$ff : \mathcal{Mod}(R) \rightarrow Vect_k^s \text{ and } ff : \mathcal{Mod}(R) \rightarrow \mathcal{Mod}(R^0) \times \mathcal{Mod}(R^0).$$

A canonical right module structure on each left  $R$  modules, say  $M$ , is given by  $mr := (-1)^{\bar{m}\bar{r}}rm$ . Now using this structure we can define tensor product of two left  $R$ - modules, say  $M_1, M_2$ , as quotient of  $M_1 \otimes_{R^0} M_2$  with submodule generated by homogeneous elements  $\{r_1 m_1 \otimes m_2 - (-1)^{\bar{m}_1} m_1 \otimes r_1 m_2 / r_1 \in R^1, m_i \in M^i\}$ . Here  $M_1 \otimes_{R^0} M_2$  is defined as a tensor product of two  $\mathbb{Z}/2\mathbb{Z}$  graded modules over a commutative ring  $R^0$ . A commutative constraint is similar to the case of tensor product of supervector spaces. Note that in general above two forgetful functors are not tensor functors. Another important notion in commutative algebra is localisation. It is easy to define localisation of rings and modules if multiplicative set is contained in centre of a ring. Hence for super commutative ring we can define localisation at any homogeneous prime ideal. It is easy to observe that given a  $R$  module  $M$  with a prime ideal  $\mathfrak{p}$ , the localisation  $M_{\mathfrak{p}} = 0$  iff  $(R^0 M)_{\mathfrak{p}} = 0$  (or  $(_{(R/J)} M)_{\mathfrak{p}} = 0$  where  $J := R \cdot R^1$ ).

## 4.2 Split Superscheme

Given any topological space  $X$  we can define super ringed space as sheaf of superring on topological space  $X$ . We shall denote sheaf of superring with  $\mathbb{Z}/2\mathbb{Z}$  grading as  $\mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1}$ . Similarly we can define sheaf of module and parity change functor  $\Pi$  over such a ringed space as before. We have following definition,

**Definition 4.2.** Given a ringed space  $(X, \mathcal{O}_X)$  is called *superspace* if ring  $\mathcal{O}_X(U)$  associated to any open subset  $U$  is supercommutative and each stalk is local ring. A *superspace* is called *superscheme* if further ringed space  $(X, \mathcal{O}_{X,0})$  is a scheme and  $\mathcal{O}_{X,1}$  is a coherent sheaf over  $\mathcal{O}_{X,0}$ .

We say that a superscheme is *affine* if the even part of structure sheaf  $(X, \mathcal{O}_{X,0})$  is affine. It is easy to see that any affine superscheme gives a super commutative ring. Equivalently an affine superscheme associated to any super commutative ring can be defined similar to usual affine scheme. Note in the definition of superscheme the odd part is coherent sheaf of module over the even part. Therefore if even part of a superscheme is noetherian then we shall get the left (or two sided) noetherian superscheme. Given a superscheme  $(X, \mathcal{O}_X)$  we can define sheaf of ideal[9]  $J_X := \mathcal{O}_X \cdot \mathcal{O}_{X,1}$ . Define  $GrX := \bigoplus_{i \geq 0} J_X^i / J_X^{i+1}$  where  $J_X^0 := \mathcal{O}_X$  and we denote the first term of  $GrX$  as  $Gr_0X = \mathcal{O}_X / J_X$ . Now using these notation we can define structure sheaves of *even* scheme and *reduced* scheme associated to superscheme  $X$  as follows,

$$\mathcal{O}_{X_{rd}} := Gr_0X \text{ and } \mathcal{O}_{X_{red}} := \mathcal{O}_X / \sqrt{J_X}.$$

Here  $J_X / J_X^2$  is a locally free sheaf of finite rank  $0|d$  for some  $d$  over  $\mathcal{O}_{X_{rd}}$ . And  $GrX$  is a Grassmann algebra over  $\mathcal{O}_{X_{rd}}$  of locally free sheaf  $J_X / J_X^2$ . Following particular class of superschemes are defined in paper of Manin[9].

**Definition 4.3.** A superscheme  $(X, \mathcal{O}_X)$  is called *split* if the graded sheaf  $GrX$  with mod 2 grading is isomorphic as a locally superringed sheaf to structure sheaf  $\mathcal{O}_X$ .



Manin[9] had also given a way to construct such a *split* superscheme. If we take purely even scheme  $(X, \mathcal{O}_X)$  and a locally free sheaf  $\mathcal{V}$  over  $\mathcal{O}_X$  then we can define symmetric algebra of odd locally free sheaf  $\Pi\mathcal{V}$ , which is denoted  $S(\Pi\mathcal{V})$  (see Manin[9]), then  $(X, S(\Pi\mathcal{V}))$  is a split superscheme. An important example is given by projective superscheme  $\mathbb{P}^{m|n}$  which is given by locally free sheaf  $\mathcal{O}(-1)^n$ . An example of a nonsplit superscheme given in Manin[9] is Grassmann superscheme  $G(1|1, \mathbb{C}^{2|2})$  which is also an example of non super-projective scheme. We can define an abelian category of sheaf of left modules over  $\mathcal{O}_X$ , denoted  $\mathcal{M}\mathfrak{od}^s(X)$  or  $\mathcal{M}\mathfrak{od}(\mathcal{O}_X)$ . As above we have a natural right module structure given by the Koszul sign rule. When  $(X, \mathcal{O}_X)$  is affine superscheme given by super ring  $R$  then we can define the sheaf of module associated to any  $R$ -module  $M$  similar to commutative case. Hence we can define quasi-coherent and coherent sheaves over any superscheme. Therefore we shall get two abelian subcategories namely category of all quasi-coherent sheaves and coherent sheaves. We denote them by  $\mathcal{Q}\mathcal{C}\mathfrak{oh}(\mathcal{O}_X)$  and  $\mathcal{C}\mathfrak{oh}(\mathcal{O}_X)$  respectively. Now similar to above we have forgetful functor as follows,

$$ff : \mathcal{M}\mathfrak{od}(\mathcal{O}_X) \rightarrow \mathcal{M}\mathfrak{od}(\mathcal{O}_{X,0}) \times \mathcal{M}\mathfrak{od}(\mathcal{O}_{X,0}).$$

It is an exact faithful functor. we can easily see that

$$\begin{aligned} \mathcal{Q}\mathcal{C}\mathfrak{oh}(\mathcal{O}_X) &= ff^{-1}(\mathcal{Q}\mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0}) \times \mathcal{Q}\mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0})) \\ \mathcal{C}\mathfrak{oh}(\mathcal{O}_X) &= ff^{-1}(\mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0}) \times \mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0})). \end{aligned}$$

We can define the tensor product of two sheaves of modules over superscheme similar to usual scheme. We shall use the canonical identification of sheaf of left and right modules by Koszul sign rule. Define tensor product of two sheaves of modules  $\mathcal{F}_1$  and  $\mathcal{F}_2$  as the sheaf associated to pre sheaf given by

$$U \mapsto (\mathcal{F}_1 \otimes \mathcal{F}_2)(U) := \mathcal{F}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}_2(U).$$

Note that with this definition of tensor structure the commutative constraint is given by sign rule i.e.  $(-1) : \Pi\mathcal{F} \otimes \Pi\mathcal{G} \rightarrow \Pi\mathcal{G} \otimes \Pi\mathcal{F}$  for purely odd sheaves and identity for other sheaves. Now we can prove some easy properties of this tensor product by just reducing to affine case,

**Proposition 4.4.** *Suppose  $(X, \mathcal{O}_X)$  is a split superscheme and  $\mathcal{F}$  and  $\mathcal{G}$  are quasi coherent sheaves.*

1. *Any  $\mathcal{O}_{X_{rd}}$  quasi coherent sheaf  $\mathcal{F}^0$  is also a  $\mathcal{O}_X$  quasi coherent sheaf via canonical projection  $\mathcal{O}_X \rightarrow \mathcal{O}_{X_{rd}}$ . Hence we get a functor  $\mathbf{i}_{rd} : \mathcal{D}_{qc}(X_{rd}) \rightarrow \mathcal{D}_{qc}(X)$ .*
2. *The functor  $\mathbf{i}_{rd}$  is a tensor functors and the images of this functor is tensor ideals in  $\mathcal{D}_{qc}(X)$ . The functor  $\mathbf{i}_{rd}$  is in fact a dominant tensor functor.*
3.  *$(\Pi\mathcal{F}) \otimes \mathcal{G} = \mathcal{F} \otimes (\Pi\mathcal{G}) = \Pi(\mathcal{F} \otimes \mathcal{G})$ .*

*Proof.* The proofs of 1 and 3 are clear from the definition. Hence we just indicate the proof of 2.

*Proof of 2.* Given any quasi coherent sheaf  $\mathcal{F}$ , observe that  $\mathbf{i}_{rd}(\mathcal{F})$  has the trivial action of ideal sheaf  $J_X$ . Therefore by definition of tensor product it follows that  $\mathbf{i}_{rd}$  is a tensor functor. Also observe that given a sheaf of  $\mathcal{O}_X$  module,  $\mathcal{F}$ , the

tensor  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}}$  has the trivial action of the ideal sheaf  $J_X$  and hence it will be in the image of the functor  $\mathbf{i}_{rd}$ . Since  $(X, \mathcal{O}_X)$  is a split superscheme, we have identification of  $\mathcal{O}_X$  with  $GrX$ . The sheaf  $GrX$  is an exterior algebra over purely odd locally free sheaf  $\Pi\mathcal{V} := J_X/J_X^2$  and each subquotients  $J_X^i/J_X^{i+1}$  can be identifies with  $\Pi^i \Lambda^i \mathcal{V}$ . Hence each subquotients are purely odd or purely even locally free sheaves. The  $\mathbb{Z}$ -grading on sheaf  $GrX$  gives a filtration for structure sheaf  $\mathcal{O}_X$  and hence we have following Postnikov tower for each complex of quasi coherent sheaf  $\mathcal{F}$ ,

$$\begin{array}{ccccccc} \mathcal{F} & \xleftarrow{\quad} & J_X \otimes \mathcal{F} & \cdots & J_X^{n-1} \otimes \mathcal{F} & \xleftarrow{\quad} & J_X^n \otimes \mathcal{F} \\ & \searrow & \nearrow & & \searrow & \nearrow & \searrow \\ \mathcal{O}_{X_{rd}} \otimes \mathcal{F} & & & \cdots & \Pi^{n-1} \Lambda^{n-1} \mathcal{V} \otimes \mathcal{F} & & \Pi^n \Lambda^n \mathcal{V} \otimes \mathcal{F} \end{array}$$

In above tower the lower order term is complex of either purely odd or purely even sheaves. And using  $\mathcal{B}$ , we have  $\Pi^i \Lambda^i \mathcal{V} \otimes \mathcal{F} = (\Pi^i \mathcal{O}_{X_{rd}}) \otimes (\Lambda^i \mathcal{V} \otimes \mathcal{F})$ . Therefore the ideal generated by the image of the functor  $\mathbf{i}_{rd}$  contains the all lower order term of above Postnikov tower and hence  $\mathbf{i}_{rd}$  is a dominant tensor functor.  $\square$

Given a split superscheme  $(X, \mathcal{O}_X = S \cdot (\Pi\mathcal{V}) = \Pi\Lambda \cdot (\mathcal{V}))$  there is one more forgetful functor as follows,

$$ff : \mathcal{Mod}(\mathcal{O}_X) \rightarrow \mathcal{Mod}(\mathcal{O}_{X_{rd}}) \times \mathcal{Mod}(\mathcal{O}_{X_{rd}}).$$

This functor is defined using the obvious inclusion of  $\mathcal{O}_{X_{rd}}$  inside  $\mathcal{O}_X$  which comes from the definition of split superscheme. For simplicity take  $\Lambda^\sharp := \Lambda^\sharp(\mathcal{V})$  for  $\sharp = ev$  or  $odd$ . Suppose  $\eta : \Lambda^\sharp \otimes \Lambda^\flat \rightarrow \Lambda^{\sharp+\flat}$ , where tensoring is over  $\mathcal{O}_{X_{rd}}$  which we omit writing later also and  $\sharp + \flat$  is defined evidently, and where  $\sharp = ev$  or  $odd$  and  $\flat = ev$  or  $odd$ , represents the natural multiplication of subsheaves of Grassmann algebra. Given a  $\mathcal{O}_X$  module  $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$  we have multiplication structure of  $\mathcal{O}_X$  which can be described using following maps,

$$m_i^{ev} : \Lambda^{ev} \otimes \mathcal{F}^i \rightarrow \mathcal{F}^i \text{ and } m_i^{odd} : \Lambda^{odd} \otimes \mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \text{ where } i \in \mathbb{Z}/2\mathbb{Z}, \quad (1)$$

and certain commutative diagrammes, where  $i \in \mathbb{Z}/2\mathbb{Z}$ ,

$$\begin{array}{ccc} \Lambda^{ev} \otimes \Lambda^{ev} \otimes \mathcal{F}^i & \xrightarrow{1 \otimes m_i^{ev}} & \Lambda^{ev} \otimes \mathcal{F}^i \\ \eta \otimes 1 \downarrow & & \downarrow m_i^{ev} \\ \Lambda^{ev} \otimes \mathcal{F}^i & \xrightarrow{m_i^{ev}} & \mathcal{F}^i \end{array} \quad \begin{array}{ccc} \Lambda^{odd} \otimes \Lambda^{odd} \otimes \mathcal{F}^i & \xrightarrow{1 \otimes m_i^{odd}} & \Lambda^{odd} \otimes \mathcal{F}^{i+1} \\ \eta \otimes 1 \downarrow & & \downarrow m_{i+1}^{odd} \\ \Lambda^{ev} \otimes \mathcal{F}^i & \xrightarrow{m_i^{ev}} & \mathcal{F}^i \end{array}$$

$$\begin{array}{ccc} \Lambda^{ev} \otimes \Lambda^{odd} \otimes \mathcal{F}^i & \xrightarrow{1 \otimes m_i^{odd}} & \Lambda^{ev} \otimes \mathcal{F}^{i+1} \\ \eta \otimes 1 \downarrow & & \downarrow m_{i+1}^{ev} \\ \Lambda^{odd} \otimes \mathcal{F}^i & \xrightarrow{m_i^{odd}} & \mathcal{F}^{i+1} \end{array} .$$

It is now easy to see that pair of sheaves  $(\mathcal{F}^0, \mathcal{F}^1)$  of  $\mathcal{O}_{X_{rd}}$  modules with maps and commutative diagrammes as above will give the sheaf of  $\mathcal{O}_X$  module structure on  $\mathcal{F} := \mathcal{F}^0 \oplus \mathcal{F}^1$ . Using property of forgetful functor we can prove that

any quasi-coherent (resp. coherent) sheaf of  $\mathcal{O}_X$  module comes from a pair of quasi coherent (resp. coherent) sheaves of  $\mathcal{O}_{X_{rd}}$  modules with the data of maps and diagrammes as above1. Note also that the Grassmann algebra constructed from locally free sheaf  $\mathcal{V}$  gives a locally free sheaf of  $\mathcal{O}_{X_{rd}}$  module. Therefore structure sheaf  $\mathcal{O}_X$  is locally free sheaf as a  $\mathcal{O}_{X_{rd}}$  module.

Similar to usual scheme we can take  $\mathcal{D}(X) := \mathcal{D}(\mathcal{Mod}(X))$  the derived category of abelian category  $\mathcal{Mod}(X)$ . There are various triangulated subcategories like  $\mathcal{D}_{qc}^\sharp(X) := \mathcal{D}^\sharp(\mathcal{QCoh}(X))$  and  $\mathcal{D}_{coh}^\sharp(X) := \mathcal{D}^\sharp(\mathcal{Coh}(X))$  where  $\sharp = +, -, b$  or  $\emptyset$ . For convenience we shall denote by  $\mathcal{D}^\sharp(X^0) := \mathcal{D}^\sharp(\mathcal{Mod}(\mathcal{O}_X^0))$  (resp.  $\mathcal{D}^\sharp(X_{rd}) := \mathcal{D}^\sharp(\mathcal{Mod}(\mathcal{O}_{X_{rd}}))$ ) the derived category of modules over purely even scheme  $(X, \mathcal{O}_X^0)$  (resp.  $X_{rd} = (X, Gr_0 X)$ ). Similar notation we can have for the other subcategories. We shall now define the another important triangulated subcategory of  $\mathcal{D}_{qc}(X)$ .

**Definition 4.5.** Given a complex  $\mathcal{F}^\cdot$  of quasi coherent sheaves of modules over superscheme  $(X, \mathcal{O}_X)$  is called *strictly perfect* if  $\mathcal{F}^\cdot$  is quasi isomorphic to bounded complex of locally free coherent sheaf of  $\mathcal{O}_X$  module. A complex  $\mathcal{F}^\cdot$  is called *perfect* if it is locally quasi isomorphic to bounded complex of locally free coherent sheaves.

We shall denote the triangulated subcategory of all perfect complexes as  $\mathcal{D}^{per}(X) \subseteq \mathcal{D}_{qc}(X)$ . Similar to scheme case we can extend various functors at the level of these triangulated categories. Hence we can prove  $\mathcal{D}^{per}(X)$  is a tensor triangulated category with tensor given by derived functor of usual tensor product defined as above. We can extend the forgetful functor defined earlier using exactness,

$$ff : \mathcal{D}^\sharp(X) \rightarrow \mathcal{D}^\sharp(X^0) \times \mathcal{D}^\sharp(X^0) \text{ and } ff : \mathcal{D}_{qc}^\sharp(X) \rightarrow \mathcal{D}_{qc}^\sharp(X^0) \times \mathcal{D}_{qc}^\sharp(X^0).$$

Here  $\sharp \in \{+, -, b, \emptyset\}$ . We can have similar forgetful functors for the case of coherent sheaves. If we restrict to split superscheme then we can also have forgetful functor for the case of locally free sheaves (or vector bundles). Hence for a split superschemes we have following forgetful functor for the triangulated subcategory of perfect complexes,

$$ff : \mathcal{D}^{per}(X) \rightarrow \mathcal{D}^{per}(X_{rd}) \times \mathcal{D}^{per}(X_{rd})$$

Note that this functor may not be a tensor functor.

### 4.3 Main Results

Now we have following result which gives a way to get back quasi coherent complexes over superscheme with a pair of quasi coherent complexes over *purely even superscheme* (or *usual scheme*).

**Lemma 4.6.** *Given a split superscheme  $(X, \mathcal{O}_X)$ , take two quasi coherent complexes  $\mathcal{F}^0$  and  $\mathcal{F}^1$  over purely even superscheme  $X_{rd}$ . Suppose we have maps  $m_i^{ev} : \Lambda^{ev} \otimes \mathcal{F}^i \rightarrow \mathcal{F}^i$  and  $m_i^{odd} : \Lambda^{odd} \otimes \mathcal{F}^i \rightarrow \mathcal{F}^{i+1}$ , where  $i \in \mathbb{Z}/2\mathbb{Z}$ , with commutative diagrammes as before1 then  $\mathcal{F} := \mathcal{F}^0 \oplus \mathcal{F}^1$  has a structure of quasi coherent complex over  $X$ .*

*Proof.* Since  $\mathcal{V}$  is locally free sheaf therefore giving a multiplication structure at the level of complexes is same as giving a complex of quasi coherent sheaves with such a multiplication structure.  $\square$

Now similar to usual scheme we can define *support* of a quasi coherent sheaf as a closed subset of  $X$  containing all super prime ideals where stalk of the sheaf is nonzero. Since nontriviality of a stalk at any point  $\mathfrak{p}$  is a local property we can check it in an affine open set containing  $\mathfrak{p}$ . Now from earlier observation  $\mathcal{F}_{\mathfrak{p}} = 0$  iff  $\mathcal{F}_{\mathfrak{p}}^0 = 0 = \mathcal{F}_{\mathfrak{p}}^1$  as a stalk of a sheaf of  $\mathcal{O}_{X_{rd}}$  modules  $\mathcal{F}^0$  and  $\mathcal{F}^1$ . Therefore for a quasi coherent sheaf  $\mathcal{F}$  we have  $\text{supp}(\mathcal{F}) = \text{supp}(ff(\mathcal{F})) = \text{supp}(\mathcal{F}^0) \cup \text{supp}(\mathcal{F}^1)$ . Now the assignment of support can be extended to derived category as follows,

$$\text{supph}(\mathcal{F}^\bullet) := \cup_{i \in \mathbb{Z}} \mathcal{H}^i(\mathcal{F}^\bullet).$$

This association can be restricted to thick subcategory  $\mathcal{D}^{per}(X)$ . As forgetful functor is an exact functor we have following relation of support similar to sheaf case,

$$\text{supph}(\mathcal{F}^\bullet) = \text{supph}(ff(\mathcal{F}^\bullet)) = \text{supph}(\mathcal{F}^0) \cup \text{supph}(\mathcal{F}^1)$$

Using this property of support we can prove following result,

**Lemma 4.7.** *The pair  $(X, \text{supph})$  defined as above gives a support data on a triangulated category  $\mathcal{D}^{per}(X)$ .*

*Proof.* Since forgetful functor is an exact functor and we have equality  $(\mathcal{F}^\bullet) = \text{supph}(ff(\mathcal{F}^\bullet))$  therefore the support data properties (SD 1)-(SD 4)[2] are easy to prove. We shall just prove (SD 5) here. Again checking nontriviality of stalk is a local question, we can assume that  $X$  is an affine superscheme. First we observe that any perfect complex  $\mathcal{F}^\bullet$  is just a strict perfect and hence bounded complex of projective modules. Therefore using induction on lengths of complexes we can reduce to proving the statement for a modules  $M$  and  $N$ . Since trivially  $M_{\mathfrak{p}} = 0$  or  $N_{\mathfrak{p}} = 0$  gives  $M_{\mathfrak{p}} \otimes N_{\mathfrak{p}} = 0$ , it is enough to prove  $M_{\mathfrak{p}} \neq 0$  and  $N_{\mathfrak{p}} \neq 0 \Rightarrow M_{\mathfrak{p}} \otimes N_{\mathfrak{p}} \neq 0$ . But by taking two elements  $m \in M$  and  $n \in N$  with  $\text{ann}(m) \cap (R - \mathfrak{p}) = \text{ann}(n) \cap (R - \mathfrak{p}) = \emptyset$  we can easily see that  $\text{ann}(m \otimes n) \cap (R - \mathfrak{p}) = \emptyset$ . Hence we have  $M_{\mathfrak{p}} \otimes N_{\mathfrak{p}} \neq 0$  and this will prove (SD 5).  $\square$

We shall prove now that above support data is in fact a classifying support data as defined in Balmer[2]. We need following classification of thick tensor subcategories[2] of  $\mathcal{D}^{per}(X)$  which we prove by relating it with the usual scheme case.

**Proposition 4.8.** *Given a split superscheme  $(X, \mathcal{O}_X)$  we have a following bijection,*

$$\theta : \{Y \subset X | Y \text{ specialisation closed}\} \xrightarrow{\sim} \{\mathcal{I} \subset \mathcal{D}^{per}(X) | \mathcal{I} \text{ radical thick } \otimes\text{-ideal}\}$$

*defined by  $Y \mapsto \{\mathcal{F}^\bullet \in \mathcal{D}^{per}(X) | \text{supph}(\mathcal{F}^\bullet) \subset Y\}$ , with inverse, say  $\eta$ ,  $\mathcal{I} \mapsto \text{supph}(\mathcal{I}) := \cup_{\mathcal{F}^\bullet \in \mathcal{I}} \text{supph}(\mathcal{F}^\bullet)$ .*

*Proof.* Using support data properties (SD 1) - (SD 5) we can prove that  $\theta(Y)$  is a radical thick tensor ideal and hence the map  $\theta$  is well defined. To prove that  $\eta(\mathcal{I})$  is a specialisation closed subset it is enough to prove that for any  $y \in \eta(\mathcal{I})$  there is a closed set containing this point. By definition  $y$  is in homological support of some object  $\mathcal{F}^\bullet \in \mathcal{I}$ . Hence  $y \in \text{supph}(ff(\mathcal{F}^\bullet))$  which is a closed subset.

It is easy to check that  $\eta \circ \theta(Y) \subseteq Y$  and  $\mathcal{I} \subseteq \theta \circ \eta(\mathcal{I})$ . To prove that  $Y \subseteq \eta \circ \theta(Y)$

it is enough to say that for any closed subset  $Z$  there exists an object with support  $Z$ . But there exists a  $\mathcal{O}_{X_{rd}}$  perfect sheaf with support  $Z$  and hence via natural map  $\mathcal{O}_X \rightarrow \mathcal{O}_{X_{rd}}$  we get a perfect sheaf with support  $Z$ .

Finally to prove that  $\theta \circ \eta(\mathcal{I}) \subseteq \mathcal{I}$  it is enough to prove that for any  $\mathcal{F} \in \theta \circ \eta(\mathcal{I})$  the object  $\mathcal{F} \in \mathcal{I}$ . Now following proof of theorem 3.15 of Thomason[17] it reduces to proving that  $\text{supph}(\mathcal{F}) \subseteq \text{supph}(\mathcal{G})$  for some object of  $\mathcal{G} \in \mathcal{I}$  then  $\mathcal{F} \in \mathcal{I}$ . But  $\mathcal{F} \otimes \mathcal{O}_{X_{rd}}$  will be in thick tensor ideal generated by  $\mathcal{G} \otimes \mathcal{O}_{X_{rd}}$  as there is a dominant tensor inclusion of  $\mathcal{D}^{per}(X_{rd})$  in  $\mathcal{D}^{per}(X)$  4.4. This will prove  $\mathcal{F} \otimes \mathcal{O}_{X_{rd}} \in \mathcal{I}$ . But  $\mathcal{I}$  is intersection of prime ideal containing  $\mathcal{I}$  and  $\mathcal{O}_{X_{rd}}$  is not in any prime ideal and hence  $\mathcal{F} \in \mathcal{I}$   $\square$

With this result it follows that  $(X, \text{supph})$  is a classifying support data on a tensor triangulated category  $\mathcal{D}^{per}(X)$  as other properties mentioned in Balmer[2] is clearly holds for  $X$ . Using Theorem 5.2 of Balmer[2] we get following corollary,

**Corollary 4.9.** *The canonical map  $f : X \rightarrow \text{Spc}(\mathcal{D}^{per}(X))$  given by  $x \mapsto \{\mathcal{F} \in \mathcal{D}^{per}(X) | x \notin \text{supph}(\mathcal{F})\}$  is a homeomorphism.*

Now we shall prove the localisation theorem similar to Thomason for split superscheme case by using the generalisation of Thomason result proved by Neeman[13]. First we recall some notations. Given a closed subset  $Z$  of  $X$  we can define full triangulated subcategory  $\mathcal{D}_{qc,Z}(X) \subseteq \mathcal{D}_{qc}(X)$  containing all objects with homological support contained in closed subset  $Z$ . Suppose  $U$  is an open complement of closed subset  $Z$ . There is a canonical restriction functor  $j^* : \mathcal{D}_{qc}(X) \rightarrow \mathcal{D}_{qc}(U)$  and clearly it will be trivial functor on thick subcategory  $\mathcal{D}_{qc,Z}(X)$ . Now using forgetful functor we have following commutative diagram,

$$\begin{array}{ccccc} \mathcal{D}_{qc,Z}(X) & \xrightarrow{\quad} & \mathcal{D}_{qc}(X) & \xrightarrow{j^*} & \mathcal{D}_{qc}(U) \\ \downarrow ff & & \downarrow ff & & \downarrow ff \\ \mathcal{D}_{qc,Z}(X_{rd}) \times \mathcal{D}_{qc,Z}(X_{rd}) & \twoheadrightarrow & \mathcal{D}_{qc}(X_{rd}) \times \mathcal{D}_{qc}(X_{rd}) & \twoheadrightarrow & \mathcal{D}_{qc}(U_{rd}) \times \mathcal{D}_{qc}(U_{rd}). \end{array}$$

We have following result which we shall prove using usual scheme case as before,

**Proposition 4.10.** *The canonical functor induced from the functor  $j^*$ , say  $j^* : \mathcal{D}_{qc}(X)/\mathcal{D}_{qc,Z}(X) \xrightarrow{\sim} \mathcal{D}_{qc}(U)$  is an equivalence.*

*Proof.* We shall prove that  $j^*$  is fully faithful and essentially surjective. Recall that the quotient functor  $j^*$  gives a map between morphisms as follows,

$$[\mathcal{F} \leftarrow^s \mathcal{F}' \rightarrow^a \mathcal{G}] \mapsto [j^* \mathcal{F} \leftarrow^{j^*s} j^* \mathcal{F}' \rightarrow^{j^*a} j^* \mathcal{G}].$$

Now using the forgetful functor we can get a similar map between morphisms of  $\mathcal{O}_{X_{rd}}$  perfect complexes.

To prove faithfulness suppose there exists a morphism  $\tilde{t} : \tilde{\mathcal{F}}'' \rightarrow \mathcal{F}'$  with  $a \circ \tilde{t} = 0$  and cone of  $\tilde{t}$  is an object of  $\mathcal{D}_{qc,Z}(X)$ . Since we have equivalence of functor  $j^*$  after applying forgetful functor therefore there exists an object  $\mathcal{F}' := \mathcal{F}''^0 \oplus \mathcal{F}''^1$  and a map  $t := t^0 \oplus t^1 : \mathcal{F}'' \rightarrow \mathcal{F}'$  with  $ff(a) \circ t = 0$ . Now using lemma 4.6 it is enough to prove that  $\mathcal{F}''$  has multiplication structure and the map  $t$  is

compatible with it. Again using the fullness of  $ff(j^*)$ , the multiplicative structure on  $\tilde{\mathcal{F}}''$  can be lifted to  $\mathcal{F}''$ . Now using faithfulness of  $ff(j^*)$  we get the commutativity of various diagram for multiplicative structure. Since  $\tilde{t}$  is compatible with multiplicative structure on  $\tilde{\mathcal{F}}''$ , again faithfulness gives compatibility of the multiplicative structure on  $\mathcal{F}''$  with the map  $t$ . Since support doesn't change under forgetful functor we get the cone of  $t$  as an object of  $\mathcal{D}_{qc,Z}(X)$  and  $a \circ t = 0$ .

We shall use similar idea as above to prove fullness of the functor  $j^*$ . So given any map  $[j^* \mathcal{F} \xleftarrow{\tilde{s}} \tilde{\mathcal{F}}' \xrightarrow{\tilde{a}} j^* \mathcal{G}]$  we want it to be image of some map  $[\mathcal{F} \xleftarrow{s} \mathcal{F}' \xrightarrow{a} \mathcal{G}]$  under  $j^*$ . But there exists a map  $[ff(\mathcal{F}) \xleftarrow{s} \mathcal{F}'^0 \oplus \mathcal{F}'^1 \xrightarrow{a} ff(\mathcal{G})]$  which maps to above map via  $ff(j^*)$ . As above using lemma 4.6 it is enough to give multiplicative structure on  $\mathcal{F}' := \mathcal{F}'^0 \oplus \mathcal{F}'^1$  which is compatible with maps  $s$  and  $a$ . Now using fullness of the functor  $ff(j^*)$  we can prove existence of multiplication map and using faithfulness we can see commutativity of various diagram to lift multiplicative structure from  $\tilde{\mathcal{F}}'$ . Once again using faithfulness of  $ff(j^*)$  we can lift the compatibility of multiplicative structure with the maps  $s$  and  $a$ .

Now to prove essential surjectivity of functor  $j^*$ , we start with an object  $\tilde{\mathcal{F}} \in \mathcal{D}_{qc}(U)$ . Since  $ff(j^*)$  is essentially surjective we get an isomorphism of  $ff(\tilde{\mathcal{F}})$  with an object  $\tilde{\mathcal{F}}'^0 \oplus \tilde{\mathcal{F}}'^1$  which is an image of the object  $\mathcal{F}'^0 \oplus \mathcal{F}'^1$  via the functor  $ff(j^*)$ . We can give the multiplicative structure on  $\tilde{\mathcal{F}}'^0 \oplus \tilde{\mathcal{F}}'^1$  s.t it becomes isomorphic to  $\tilde{\mathcal{F}} \in \mathcal{D}_{qc}(U)$ . Now we can lift this multiplicative structure to  $\mathcal{F}' := \mathcal{F}'^0 \oplus \mathcal{F}'^1$  using fully faithfulness of  $ff(j^*)$ . Hence  $j^*$  is essentially surjective.  $\square$

Another notion which we need is compact object in a triangulated category and compactly generated triangulated category.

**Definition 4.11.** (a) An object  $t$  in a triangulated category, which is closed under formation of every small coproducts, is said to be *compact* if  $Hom(t, \bigoplus_i x_i) \cong \bigoplus_i Hom(t, x_i)$  respects coproducts. In a triangulated category  $\mathcal{T}$ , the full subcategory of all compact objects is denoted as  $\mathcal{T}^c$ .

(b) A triangulated category  $\mathcal{T}$ , which is closed under formation of every small coproducts, is said to be *compactly generated* if there exists a small set  $T$  of compact objects s.t.  $\mathcal{T}$  is a smallest triangulated subcategory closed under coproducts and distinguished triangles containing  $T$ . Equivalently,  $\mathcal{T}$  is called *compactly generated* iff  $T^\perp := \{x \in \mathcal{T} | Hom_{\mathcal{T}}(t, x) = 0 \text{ for all } t \in T\} = 0$ . The set of compact objects  $T$  is called *generating set* if further  $T$  is closed under suspension or translation.

Now we shall recall the theorem 2.1 of Neeman[14] which is proved in quite generality and is a slight strengthening of theorem 2.1 of Neeman[13].

**Theorem 4.12** (Neeman[13][14]). *Let  $\mathcal{S}$  be a compactly generated triangulated category. Let  $\mathcal{R}$  be set of compact objects of  $\mathcal{S}$  closed under suspension and  $\mathcal{R}$  be a localising subcategory generated by  $\mathcal{R}$  in triangulated category  $\mathcal{S}$ . Under these hypothesis[13] there exists the quotient category  $\mathcal{T}$  of  $\mathcal{S}$  with adjoint functor of natural functor  $j^*$  i.e. there is following sequence of triangulated categories,*

$$\mathcal{R} \rightarrow \mathcal{S} \xrightarrow{j^*} \mathcal{T}.$$

It induces a functors at the level of subcategories of compact objects i.e.

$$\mathcal{R}^c \rightarrow \mathcal{S}^c \xrightarrow{j^*} \mathcal{T}^c.$$

1. The category  $\mathcal{R}$  is compactly generated, with  $R$  as a generating set.
2. If  $R$  is generating set for all of  $\mathcal{S}$  then  $\mathcal{R} = \mathcal{S}$ .
3. If  $R \subset \mathcal{R}$  is closed under the formation of triangles and direct summands, then it is all of  $\mathcal{R}^c$ . In any case  $\mathcal{R}^c = \mathcal{R} \cap \mathcal{S}^c$ .
4. The induced functor  $F : \mathcal{S}^c / \mathcal{R}^c \rightarrow \mathcal{T}^c$  is fully faithful and every object of  $\mathcal{T}^c$  is isomorphic to direct summand of image of the functor  $F$ . In particular, if  $\mathcal{T}^c$  is an idempotent complete then we get an equivalence from idempotent completion  $\widetilde{\mathcal{S}^c / \mathcal{R}^c}$  to triangulated category  $\mathcal{T}^c$ .

In our particular situation we take  $\mathcal{S} := \mathcal{D}_{qc}(X)$ ,  $\mathcal{R} := \mathcal{D}_{qc,Z}(X)$  and as we proved above 4.10 the quotient will be  $\mathcal{T} := \mathcal{D}_{qc}(U)$ . We shall now prove following result which will provide all hypothesis for application of above theorem.

**Proposition 4.13.** *following statements are true for any split superscheme  $(X, \mathcal{O}_X)$*

1. The triangulated category  $\mathcal{D}_{qc}(X)$  is closed under formation of every small coproducts.
2. The triangulated category  $\mathcal{D}_{qc}(X)$  is a compactly generated category.
3.  $\mathcal{D}_{qc,Z}(X)^c \simeq \mathcal{D}_Z^{per}(X)$  for any closed subset  $Z$  of  $X$ .

*Proof.* *Proof of 1.* This is similar to usual scheme case, see example 1.3 of Neeman[14].

*Proof of 2.* Take a small set of objects  $T \subset \mathcal{D}_{qc}(X)$  of all perfect complexes of  $\mathcal{O}_{X_{rd}}$  modules via functor  $\mathbf{i}_{rd}$  and its full image under the functor  $\Pi$ . Now using the canonical filtration and result 4.4 we have following Postnikov tower for every object  $\mathcal{F} \in \mathcal{D}_{qc}(X)$ ,

$$\begin{array}{ccccccc} \mathcal{F} & \xleftarrow{\quad} & \mathcal{G}^1 & \cdots & \mathcal{G}_{n-1} & \xleftarrow{\quad} & \mathcal{G}_n \cdots \\ & \searrow & \nearrow & & \searrow & \nearrow & \\ & \mathcal{F}_1 & & \cdots & \mathcal{F}_{n-1} & & \mathcal{F}_n \end{array}$$

Now the base of above tower,  $\mathcal{F}_i := \mathcal{F} \otimes_{\mathcal{O}_X} \Pi^i \Lambda^i(\mathcal{V}) \in Im(\mathbf{i}_{rd})$ , is generated by objects of the set  $T$  and hence every object  $\mathcal{F} \in \mathcal{D}_{qc}(X)$  is generated by the set  $T$ .

*Proof of 3.* It is enough to prove that all perfect complexes are compact objects. Indeed, the full subcategory of perfect complexes is closed under triangles and direct summands similar to usual scheme. Hence by taking  $R$  to be all perfect complexes the above result of Neeman 4.12 proves that all compact objects are perfect complexes. Now to prove that every perfect complex is compact object we have to first observe following,

$$(H^0(\mathcal{R}Hom(\mathcal{F}, \mathcal{G})))^0 = Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Here  $\mathcal{R}Hom(\mathcal{F}, \mathcal{G})$  is an internal homomorphisms between  $\mathcal{F}$  and  $\mathcal{G}$ . Now rest of the proof is similar to the proof given in example 1.13 of Neeman[14].  $\square$



Using the above result 4.12 it is easy to deduce following corollary,

**Corollary 4.14.** *Given a split superscheme  $(X, \mathcal{O}_X)$  we have an equivalence of tensor triangulated categories,  $F : \widetilde{\mathcal{D}^{per}(X)} / \mathcal{D}_Z^{per}(X) \xrightarrow{\sim} \mathcal{D}^{per}(U)$ .*

*Proof.* It is enough to observe that  $j^*$  induces a tensor functor.  $\square$

Similar to Balmer[2] we shall use above localisation result to give relation between structure sheaves. Balmer[2] had defined structure sheaf of  $\mathrm{Spc}(\mathcal{K})$  for any tensor triangulated category as a sheaf associated to the presheaf given by  $U \mapsto \mathrm{End}_{\mathcal{K}/\mathcal{K}_Z}(1_U)$  where  $U$  is an open set and  $1_U \in (\mathcal{K}/\mathcal{K}_Z)$  is the image of tensor unit  $1 \in \mathcal{K}$ . Define  $\mathrm{Spec}(\mathcal{D}^{per}(X)) := (\mathrm{Spc}(\mathcal{D}^{per}(X)), \mathcal{O}_{\mathcal{D}^{per}(X)})$  the locally ringed space associated to tensor triangulated category  $\mathcal{D}^{per}(X)$ . Now the homeomorphism  $f$  4.9 defined above for split superscheme gives a map of locally ringed space,  $f : (X \simeq X^0, \mathcal{O}_{X^0}) \rightarrow \mathrm{Spec}(\mathcal{D}^{per}(X))$ . Here the map of structure sheaves comes from the identification given in corollary 4.14. We have following result similar to Theorem 6.3 of Balmer[2],

**Theorem 4.15.** *Suppose  $X$  is a topologically noetherian (that is, if all open subset are quasi compact) split superscheme. The map  $f$  defined as above gives an isomorphism of locally ringed space i.e.  $X^0 \simeq \mathrm{Spec}(\mathcal{D}^{per}(X))$ .*

*Proof.* Using the homeomorphism  $f$  it is enough to prove isomorphism of structure sheaves. Hence we can assume that superscheme is affine. Now using the remark 8.2 of Balmer[1] and localisation theorem 4.14 we can prove that induced map of sheaves is an isomorphism.  $\square$

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